Spinoza's Infinities, Mathematical, Logical, and Metaphysical

Joshua D. Guttman
joshua.guttman@gmail.com

June 5, 2022

1 Introduction

The puzzle is there to see on the first page of the *Ethics*. Definition 2 says:[1]

> A thing is said to be finite in its own kind when it can be limited by another thing of the same nature. For example, a body is said to be finite because we can always conceive of another body greater than it. So, too, a thought is limited by another thought. But body is not limited by thought, nor thought by body.

So *finite in its own kind* is defined by limitation by another thing of the same kind. Given the common practice of defining a term and thereby characterizing its opposite by negation, we now know that *infinite in its own kind* will mean not being limited by any other thing of the same kind.

However, only a few lines later, we find 1d6, which reads:

> By God I mean an absolutely infinite being; that is, substance consisting of infinite attributes, each of which expresses eternal and infinite essence.

This raises questions about infinity, since one wants to know what "absolutely" adds to infinite in this case; whether "infinite attributes" means infinitely *many* attributes, or perhaps it means simply all of them, or possibly serves merely to stress the point made by the third occurrence of the word "infinite" in the definition; and what makes an *essence* infinite rather than finite. Indeed, the relation between essences and limitations needs explaining. One also wants to know what it means for an attribute to "express essence." One suspects that every attribute expresses essence, i.e. that the nature of an attribute is to "press

[1] As usual, we will write citations to the *Ethics* in the form part-kind-number, in this case 1d2, possibly followed by c, e, or s for corollary, explication, or scholium, respectively. Since there may be more than one corollary or scholium, or occasionally, demonstration, this may again be followed by a number. All citations to Spinoza will be from Shirley's translation [23], except for a few letters from other sources. A table of contents appears on p. 26.

out" essence into determinate modes.[2] Every determination being a negation, as we learn in Letter 50 to Jelles [24], we may guess that attributes enable a kind of negativity to enter into the consequences of substance.

The reader needs an explication, and one kind of explication is fortunately immediately forthcoming (1d6e), the first sentence of which reads:

> I say 'absolutely infinite,' not 'infinite in its kind.'

But this alarms a logically minded reader, since the only notion that was already defined was in fact the phrase as a whole, "finite in its own kind." The work of this sentence is to highlight that "absolutely infinite" is not the negation of "finite in its own kind;" but then what does it mean? We next find a constraint on the answer:

> For if a thing is only infinite in its own kind, one may deny that it has infinite attributes.

Presumably space (or extension) would serve as an example here, since it is infinite in its kind, not being limited by anything of the same nature. However, it has (or rather, it *is*) only one attribute. So the claim of absolute infinity seems to require at least that all attributes are comprised, and possibly that there are infinitely many of them. A central idea follows in the next sentence:

> But if a thing is absolutely infinite, whatever expresses essence and does not involve any negation belongs to its essence.

So, absolute infinity is inclusive in that, whatever is relevant in the sense of "expressing essence" will be included within its essence, unless it would import a kind of negation. This train of thought suggests the following tentative conclusions:

Positivity appears central. All intrinsically positive characteristics would seem compatible, since they have nothing negative to contradict each other. As thoughts, they seem like they would not limit each other.

Existence seems to follow from positivity. Anything absolutely infinite would have no impediment to existing—i.e. no reason not to exist—from which its existence would follow by the Principle of Sufficient Reason.

Finiteness, on the other hand—anything limited by another—has a *negative* character, since it is not this other, and is not where this other is, and is not what this other would involve.

The word *positivity* seems appropriate because of its connection with *positing*, which arises for instance in the demonstration of 3p4:

[2]The etymology of *exprimere* from *ex-* (out) plus *primere* (to press) appears helpful. The German *ausdrucken* offers a similar underlying metaphor.

> No thing can be destroyed except by an external cause.
>
> **Demonstration.** This proposition is self-evident, for the definition of anything affirms, and does not negate, the thing's essence: that is, it posits, and does not annul, the thing's essence...

On the other hand, negativity is closely connected with finitude, as in 1p8s1:

> Since in fact to be finite is in part a negation and to be infinite is the unqualified affirmation of the existence of some nature, it follows from Proposition 7 alone that every substance must be infinite.

Contrary to usage, Spinoza has then defined the negative notion of finite rather than the intrinsically positive notion of infinity. In doing so he has perhaps made his definition more acceptable to others, who would find the mathematical and physical notion of limitation more familiar than the intrinsic positivity of the infinite. The definition of finiteness by limitation is useful also because many of Spinoza's proofs are written as proofs by contradiction. Thus, to prove something infinite, one assumes that it is finite; if no other thing of the same nature can limit it, then the desired contradiction has been found. 1p8d is an early key example of this strategy.

The goal of this paper is to study the infinite in Spinoza, clarifying its intimate connection with substance, and its differing manifestations in the modes of substance. We will focus closely on Letter 12, the Letter on the Infinite, expanding on its content with reference to the *Ethics* and a few additional letters (see especially Section 3). Ariew [2], Melamed [16, 18], and Gueroult [12] will be our main guides (Sections 5, 7). We will endeavor to explain the relations of the ideas to aspects of mathematics (Section 4) and logic (Section 6) in a pair of interludes. We have already seen that Spinoza's definitions can have a mathematical flavor, like the official definition of finiteness in 1d2, or a metaphysical flavor, as in 1d6e. However, these two senses also play out together, and do so in nature and the human, specifically in the law-like succession of the modes. Their relevance to the natural world of the modes is the focus of Section 7. An initial look at Descartes and a final glance at Leibniz (Sections 2, 8) provide framing.

2 A Backwards Glance to Descartes

Descartes, in the *Principles of Philosophy*, Part I, §25, stipulates that God is infinite, but in §§26–27, insists that in all other cases we should speak of the *indefinite*. This indefinite consists in the fact that limits cannot be discovered by us. Indeed, in Part II, §21, we learn that wherever, in the totality of corporeal substance, "we imagine a limit, we are not only still able to imagine beyond that limit spaces indefinitely extended, but we perceive these to be in reality such as we imagine them." ([7, ed. Ariew, p. 261], AT VIIIa, 52)[3]

[3]All translations from Descartes are those of Roger Ariew's edition [7].

Thus, the indefinite is a sort of "iterative unlimitedness," in Anat Schechtman's words [22], meaning that for any number or extended quantity, a greater one exists. By contrast, the infinity of God is a different matter, and a unique case. Schechtman characterizes it as an *ontic* infinity, namely an infinity that is "unqualified and equivalent to absolute independence" [22, p. 42].

Indeed, this ontic infinite of independence powers the dialectic of the *Third Meditation*, which turns on the connection between infinity and being self-caused. My weaknesses and limitations are a sure sign that I am not self-caused:

> But if I got my being from myself, I would not doubt, nor would I desire, nor would I lack anything at all. For I would have given myself all the perfections of which I have some idea.... ([7, ed. Ariew, p. 120], AT VII, 48)

Subsequently, Descartes writes of a cause,

> For if it got its existence from itself, it is evident from what has been said that it is itself God, because, having the power of existing in and of itself, it unquestionably also has the power of actually possessing all the perfections of which it has in itself an idea—that is, all the perfections that I conceive to be in God. ([7, ed. Ariew, p. 121], AT VII, 49–50)

Being a cause of itself, or Schechtman's "absolute independence," entails having infinite perfections.

Thus, there is a clear pedigree for distinguishing the infinity or indefiniteness of extension, as a sort of repeated traversal of limits, from an absolute infinity that has an ontological root in being self-caused.

Indeed, the inversion that we noted in the introduction, according to which finiteness is in fact a negation or limitation of an altogether positive infinity, can also be perceived in the *Third Meditation*:

> Nor should I think that I do not perceive the infinite by means of a true idea, but only through a negation of the finite.... On the contrary, I clearly understand that there is more reality in an infinite substance than there is in a finite one. Thus the perception of the infinite is somehow prior in me to the perception of the finite, my perception of God is prior to my perception of myself. ([7, ed. Ariew, p. 118], AT VII, 45)

The themes we identified in Spinoza grew from a seed in Descartes. Indeed, in Leibniz we have a similar pattern, which Robert Adams characterizes:

> Leibniz's conception of divine perfection commits him to agree with Descartes that, in its own nature, the divine infinity or perfection is primitive — that it is unanalyzable and not a negation of the finite. For him, as for Descartes, the infinite, in properties capable of infinity, is the primary case, and the finite is formed by limitation, or partial negation, of the infinite (NE 157f). [1, p. 116], cited in [19, p. 137]

3 Spinoza's Letter on the Infinite

Spinoza makes the infinite his theme in a letter to his friend, L. Meyer, Letter 12 dated 20 April 1663 [23, pp. 231–235], which was famous already in his own time. For instance, Leibniz is known to have read a copy of it with great care in April 1676, and to have retained a copy of his notes, as we may learn from Nachtomy [19, p. 145]. In our time, it has been carefully scrutinized by Martial Gueroult in an appendix to *Spinoza*, vol. I. [12], and more recently by Ariew [2] and Melamed [16], all from rather different points of view. The Letter on the Infinite has substantial textual overlap with a long scholium to part I of the *Ethics*, 1p15s.

Letter 12: A brief exposition (I). The content of the letter is challenging. Spinoza undertakes to explain the different types of infinity, specifically,

> what kind of infinite cannot be divided into, or possess any, parts, and what kind can be so divided without any contradiction;

as well as:

> what kind of infinite can be considered, without contradiction, as greater than another infinite, and what kind cannot be so conceived. [23, p. 232]

Spinoza warns us that we will need to distinguish among three key pairs, namely:

1. "that which must be infinite by its very nature or by virtue of its definition," as opposed to "that which is unlimited not by virtue of its essence but by virtue of its cause;"

2. "that which is called infinite because it is unlimited," as opposed to "that whose parts cannot be equated or explicated by any number, although we know its maximum and its minimum;" and finally

3. "that which we can apprehend only by intellect and not by imagination," as opposed to "that which can also be apprehended by the imagination."

He also explains *substance* and its *modes*; and that the being of substance is *eternity*, i.e. the "infinite enjoyment of its existence," while the being of modes is *duration*. Duration and the modes admit of division, whereas substance is indivisible.

Spinoza comments that *quantity* may be conceived in two ways. When conceived "abstractly or superficially, as we have it in the imagination with the help of the senses," then it is "divisible, finite, composed of parts, and multiplex." Thus, the senses and their traces in imagination are the source of this notion of decomposable quantity. Moreover, the divisible quantity results from this imaginative experience by a process of abstraction.

Spinoza contrasts it with quantity as "substance apprehended solely by means of the intellect." Although "this is very difficult," this notion of quantity is "found to be infinite, indivisible, and one alone" [23, p. 233].

On Mathematics. There follows a miniaturized exposition of a philosophy of mathematics. Here we are definitely concerned with "that which can also be apprehended by imagination," as the third opposition above has it; we recognize in the word "also" Spinoza's claim that these considerations can be apprehended not only by intellect—as doubtless everything can—but also by imagination. Thus, this link with imagination does not denigrate mathematics, but characterizes its content, as engaging intellection with the abstractions of imagination. Hence Spinoza writes in a balancing disjunction, "Measure, Time and Number are nothing other than modes of thinking, or rather, modes of imagination."

Indeed, they are modes of thinking that marshal and shape imagination. Spinoza's description of numbers makes this somewhat more concrete.

> Again, from the fact that we separate the Affections of Substance from Substance itself, and arrange them in classes so that we can form images of them as best we may, there arises Number, whereby we limit them. [23, p. 233]

We may mentally group objects into various categories, humans for instance or armadillos, although in doing so we sacrifice some of the richness of their individual existence as modes of substance. Indeed, when we group them in these ways we certainly also sacrifice the causal nexus that ties a particular human into the fabric of nature. Different humans have different positions within this causal nexus, one being say father while another is his daughter, who are, thus, linked in order in a chain of causality while remaining equally instances of humans. A scholium to *Ethics* Part II makes clear that this process of abstraction is a process of confusion:

> The human body, being limited, is capable of forming simultaneously in itself only a certain number of distinct images. ...If this number be exceeded, these images begin to be confused, and if the number of distinct images which the body is capable of forming simultaneously in itself be far exceeded, all the images will be utterly confused with one another. (2p40s1)

Thus Spinoza explains the origin of the transcendentals "entity," "thing," "something," etc. This work of confusion may nevertheless be an active, rational confusing of oneself, offering deductive power and certainty. One interprets numbers as groupings to which such a process of productive confusion—or abstraction—has been applied. The structured confusing justifies operations such as addition and subtraction, as they have corresponding effects on classes if the process of confusion identifies them as equinumerous. Letter 50 to Jelles develops a compatible view of number:

> We don't conceive things under numbers unless they have first been brought under a common genus. Someone who holds a penny and a dollar in his hand won't think of two unless he can call them by a single name such as 'coins'. When he does that, he can say that he has two coins, calling each by the name 'coin'. This shows clearly

> that a thing is called 'one' or 'unique' only after another thing has been conceived that (as they say) 'agrees with it'. [24, p. 75]

However, this rationalized abstraction from the flow of imagination has a delicate, often misunderstood status. It proceeds from the modes and their effects upon us via imagination, but the result has a quality of absoluteness and certainty. This absoluteness lies in their remoteness from the causal nexus of nature and the grounding power of substance; it requires "ignoring the efflux of Duration from things eternal," as Spinoza puts it. The phrase occurs in the first sentence of the paragraph introducing Time and Measure (p. 233). In the last sentence, Spinoza returns to a similar phrase. He writes that by confusing the "Modes of Substance" with these "beings of reason" or "aids to the imagination,"

> we are abstracting them from Substance and from the manner of their efflux from Eternity, and in such isolation they can never be correctly understood. [23, p. 233]

Hence the danger of people tying themselves "into such extraordinary knots that in the end they have been unable to extricate themselves except by breaking all laws and perpetrating the grossest absurdities;" these people are "all who have attempted to understand the workings of Nature by such concepts, and without really understanding these concepts."[4] The danger arises because "many things ... can in no way be apprehended by the imagination but only by the intellect." The attempt to understand substance by measure, time, and number is hopeless. Indeed, the "Modes of Substance," too, must not be "confused with such mental constructs (entia rationis) or aids to the imagination." Here again we have a curious disjunction between the "beings of reason" and the "aids of the imagination."

Spinoza's doctrine seems thoroughly anti-Pythagorean. Pythagoreans regard number and primitive geometric forms as prior to material reality, and as providing the rationale according to which it is constructed, or even the material out of which it is constructed. Plato's *Timaeus* is presumably the historically most influential document of this view [21, St. 31ff]. Spinoza by contrast views the causal efflux of the modes as more fundamental, and the causal grounding of modes in substance as the most fundamental. Numbers and shapes, by contrast, are abstractions that require the activity of the imagination as an intermediate stratum. They are themselves not even ideas, as they are not even ways of conceiving modes; they are instead useful summaries of rational consequences of the confusion of our imagination.

Thus, mathematics is a meeting ground of the rational and the imaginative.

On "imagination." Spinoza explains his usage of the words "image" and "imagine" in a scholium in Book II of the *Ethics*:

[4] Perhaps Descartes's view of the material world as characterized purely by extension, and governed by laws that concern only the measure of extension, is a target of this barb.

> Further, to retain the usual terminology, we will assign the word 'images' (*imagines*) to those affections of the human body the ideas of which set forth external bodies as if they were present to us, although they do not represent shapes. And when the mind regards bodies in this way, we shall say that it 'imagines' (*imaginari*). (2p17s)

Thus, an image is an affection of the body. It may be mirrored (cf. 2p7) in an idea that represents (as its intentional object) this affection of the body. Since these affections of the body are the effects of causes in extended nature, our corresponding ideas are related to those causes also, which the ideas may "set forth ... as if they were present to us." The activity of engaging in these ideas is imagining.

The most challenging phrase in this passage is "although they do not represent shapes." It would be attractive to interpret this phrase as distinguishing between the extended modes that cause the affections of the body, on the one hand, and the geometrical shapes that we might abstract from them. These geometrical shapes are "beings of reason" and "aids to the imagination" without being the constituents of the ideas that represent the affections of our bodies in which imagining consists. Nevertheless, the subordinate clause "although they do not represent shapes" seems a slim basis to provide much confirmation for our view that mathematical notions such as shapes are not the content of ideas in imagination, but abstractions from them.[5] We will return to the status of geometry in Section 7.1.

Spinoza continues in 2p17s, starting the analysis of error from the partiality (or inadequacy) of imagination. But he ends with a beautiful, deeply Spinozistic statement of the *Aufhebung* of error and non-existence into the whole and into the power of nature, expanding the thought of 2p8:

> For if the mind, in imagining non-existing things to be present to it, knew at the same time that those things did not exist in fact, it would surely impute this power of imagining not to the defect but to the strength of its own nature, especially if this faculty of imagining were to depend solely upon its own nature; that is, (Def. 7, I) if this faculty of imagining were free. (2p17s)

Where imagination and intellect interact freely in certainty, mathematics arises.

Letter 12: A brief exposition (II). The remainder of the letter draws out the consequences of the thoughts Spinoza has introduced, followed by a brief summary. That in turn is followed by a remarkable paragraph in which Hasdai

[5] Letter 50 again may provide some support:

> Someone who says that he conceives a shape is merely saying that he conceives a determinate thing and how it is determinate. So this determination is not a fact about the thing's being but its non-being. Therefore, because the shape is nothing but a determination, and determination is (as they say) a negation, it cannot be anything but a negation. [24, pp. 75–76]

Crescas gets—so to speak—the last word, which is a distinctive version of the regress argument to the existence of God. Spinoza comments:

> So the force of the argument lies not in the impossibility of an actual infinite, or an infinite series of causes, but in the assumption that things which by their own nature do not necessarily exist are not determined by a thing that necessarily exists by its own nature. [23, p. 233]

This fascinating—and richly Avicennan—conclusion sheds light on Spinoza's view of the relationship of finite modes among themselves and to the substance that underlies them. However, let us focus on the examples and consequences that form the bulk of the later portion of the letter.

First, Spinoza demystifies the "present moment," resolving a Zenonian puzzle about the passage of an hour. The flowing being of duration is rooted in the mode's inherence in substance. We experience it, as the intellect understands, as an aspect of the causal flux of nature. The appearance of paradox arises when we privilege the abstraction of time, inferring that at most some smallest part can be passing at any one time.

> Therefore, many who are not used to distinguishing mental constructs from reality have ventured to assert that Duration is composed of moments. . . . To say that Duration is made up of moments is the same as to say that Number is made by adding noughts together. [23, p. 234]

Indeed, zeros are not homogeneous with positive integers; a positive number is not the sum of any collection of zeros. In the same way, points are not homogeneous with lines. A line cannot be composed of points (cf. 1p15s), but is composed of its finite line segments.

Just as lines are homogeneous with the finite segments within them, not with points, duration is in fact composed of segments during which something is always already happening, i.e. the causal flux is in progress. Points may be regarded as the terminations of line segments, but not as their parts, a formulation one will also find in Leibniz [3]. As Cauchy was later to explain, a point is the intersection of a sequence of nested line segments, each half the width of the previous one. Thus a point is really of a different order from the line segments to which it might belong; it is the result of an infinite sequence of choices, each selecting the left or right half of its predecessor. Although Spinoza does not appear to have the terminology to express this idea directly, his views seem compatible with it.

He then argues that despite the fact that number, time, and measure, being abstractions from ideas rooted in imagination, are intrinsically finite, the actual practice of mathematicians is flexible and aware of "many things inexpressible by any number," and yet rationally controlled by them. Spinoza appears to be thinking about irrational numbers, which were systematically understood already in Euclid's *Elements*. Non-concentric circles provide another illustration of this, specifically of variations that are constrained within a minimum and maximum.

Before turning to a brief summary and ending with his comment on Hasdai Crescas, Spinoza adds:

> if anyone were to attempt to determine all the motions of matter that have ever been, by reducing them and their duration to a certain number and time, he would be attempting to deprive corporeal Substance, which we cannot conceive as other than existing, of its Affections, and bring it about that Substance should not possess the nature which it does possess. [23, p. 234]

This again appears to target Descartes's mechanistic program, which aimed to explain motion by the quantitative properties of all parts of extension. What's odd is that Spinoza speaks of depriving "corporeal Substance ... of its Affections," rather perhaps than of depriving the Affections of their underlying and animating Substance.

4 An Interlude: The mathematics of infinity

Before continuing to consider commentators on Letter 12 and on Spinoza's view of the infinite, we offer a very brief summary of the modern view of the finite and the infinite, focusing on sets and numbers. Much of the key work was done in the period from the 1820s to the 1880s. This well-defined phase of foundational work was completed in 1884–87, with the characterization of arithmetic in Frege's *The Foundations of Arithmetic* [10] and then Dedekind's *Nature and Meaning of Numbers* [6]. We will focus on Dedekind because of the compactness and clarity of his approach.

For the modern view of the real numbers and measure, the story starts with Cauchy's definition of the real numbers and various notions of continuity [14]. It progresses from Cauchy through Weierstrass, Cantor, Dedekind ("Continuity and Irrational Numbers," 1872), subsequently reaching a kind of fulfillment in Lebesgue's theory of integration (1904). We will omit this part of the story for the sake of space; it is less urgent for understanding Spinoza than for Leibniz, who was much more concerned with the how the infinite divisibility of space should be understood mathematically.

Dedekind and the definition of infinity. The core contemporary notion of infinity is due to Dedekind (1888) [6]. A function $f\colon S_1 \to S_2$ from a set S_1 to a set S_2 is said to be *injective* or *surjective* on the following conditions:

Injective: Different arguments are mapped to different values, i.e. for all $x, y \in S_1$, $x \neq y$ implies $f(x) \neq f(y)$;

Surjective: The mapping covers all of S_2, i.e. for every $z \in S_2$, there is some $x \in S_1$ such that $f(x) = z$.

A function $f: S_1 \to S_2$ is *bijective* iff it is both injective and surjective.

A Dedekind-infinite set has an injective function into a proper subset:

Dedekind-infinite: A set S is *Dedekind-infinite* iff there is an injective function $f: S \to S$ and a value $y \in S$ such that, for all $x \in S$, $f(x) \neq y$.

In this case, S has a proper subset S' such that f is a bijection between S and S'; namely, we can let $S' = \{y \in S : \exists x \in S \,.\, f(x) = y\}$, the *image* $f(S)$ of S under f. A key part of the justification for this definition is this lemma essentially due to Dedekind:

Lemma 1 *Set S is Dedekind-infinite if and only if there is an injective function $f: \mathbb{N} \to S$ from all the natural numbers $\mathbb{N}$ into distinct elements of S.*

The latter condition might previously have been considered the "canonical" definition of infinite set, but Dedekind was developing a theory of the natural numbers $\mathbb{N}$. To construct $\mathbb{N}$, he first needed a characterization of infinity that was simple and conceptually independent of $\mathbb{N}$.

To develop his theory of $\mathbb{N}$, Dedekind needs to prove that there actually is an infinite set, or "system" in his terminology. To do so, he engages in a bit of philosophy on his own account, crediting Bolzano with a similar strategy:

> **66. Theorem.** There exist infinite systems.
>
> **Proof.** My own realm of thoughts, i.e. the totality S of all things, which can be objects of my thought, is infinite.... [6, p. 64]

Dedekind argues that the set S consisting of his own realm of thoughts is infinite because the function f that maps anything s to $f(s)$, where $f(s)$ is:

> the thought s', that s can be object of my thought,

is an injective function taking values within S. Moreover, not every member of S is of the form $f(s)$; for instance, "my own ego" is not identical with the thought that any s can be object of my thought.

The definition of infinity, using the idea of an injective function whose image is a proper subset, suggests Dedekind's strategy for defining the natural numbers. Consider an infinite set S, where f is an injective function and $a_0 \in S$ is not in the image $f(S)$ of S under f. So $f(a_0) \notin f(S)$, and $f(f(a_0)) \notin f(f(S))$. So a_0 is like 0 in not being an f-successor of anything; $f(a_0)$ is like 1 in not being an f-successor of any f-successor, and so on. So the natural numbers are like the set of "finite iterates" of f, starting from a_0. But how can we characterize the "finite iterates" of f from a_0?

Here Dedekind [6] and Frege [10] hit on the same insight. The "finite iterates" of a_0 under f form the *least set* S_0 containing a_0 and closed under f. The "least set" is the *intersection* of *all* such sets, namely

$$S_0 = \bigcap\{T \subseteq S : a_0 \in T \text{ and } \forall x \,.\, x \in T \text{ implies } f(x) \in T\},$$

i.e. S_0 is the intersection of all sets $T \subseteq S$ such that $a_0 \in T$ and T is f-closed.

This definition of S_0 justifies the principle of mathematical induction, as well as the principle that we can define functions by recursion on the natural numbers. It can then be proved that the choice of $\langle S, f, a_0\rangle$ does not matter; any choice yields an isomorphic $\langle S_0, f, a_0\rangle$. Thus, we may define $\langle \mathbb{N}, \cdot + 1, 0\rangle$ by the shared isomorphism structure of all such $\langle S_0, f, a_0\rangle$.

In addition to the general value of this as clarifying the mathematician's view of infinity, there is a specific relevance to Spinoza's doctrine in Letter 12 and *Ethics* 1p15s. Spinoza writes:

> Now [mathematicians] do not draw the conclusion that it because of the multitude of parts that such things exceed all number; rather, it is because the nature of the thing is such that number is inapplicable to it without manifest contradiction. [23, p. 234]

Dedekind's procedure is faithful to this comment: He exhibits a structural property of infinite sets such as "my own realm of thoughts," which, by way of Lemma 1 above, entails that it cannot, for any $n \in \mathbb{N}$, be in bijection with the finite set of natural numbers $[1, \ldots, n]$. That is, there is a manifest contradiction in supposing it "numbered" by n. The "multitude of parts" is not part of the story.

5 Some Commentators on Letter 12

Many commentators have fastened on the Letter on the Infinite and related letters—Leibniz, for instance, already in the 1670s, and Hegel subsequently—but here we will focus on a few recent commentators, considering them in an order that meshes with our themes.

5.1 Ariew

Roger Ariew [2] relates Spinoza to Descartes and also to a scholastic tradition. He notes that Spinoza was well aware of Descartes's doctrine that God and God alone is in fact infinite, other entities being at best indefinite, for instance extension. Being indefinite is in fact a subjective condition, rooted in our being unable to determine any limits of that entity. We may even have arguments, as Descartes does for extension (*PP*, II §21 [7, p. 261]), that limits cannot be found. However, we are unable to assert a true, "ontic" infinity, to revert to Schechtman's term [22], except in God's case. Descartes also holds that our finiteness prevents us from understanding God or infinity, although we have an idea of God. This Spinoza deeply rejects, e.g. in:

> The human mind has an adequate knowledge of the eternal and infinite essence of God. *Ethics*, 2p47

Ariew further argues that Spinoza was aware—not just of a Jewish scholastic tradition including Maimonides, Gersonides, and Crescas, itself incorporating Arabic traditions including Avicenna and Averroes, which he takes Wolfson already to have established—but also of a late medieval Christian tradition including Buridan, Gregory of Rimini, and John of Bassols'.

He quotes Gregory of Rimini to interesting effect; he is cited as objecting to Peter of Spain's definition of the (categorematic) infinite as "A quantity so large that there is, and can be, no larger." Instead, he prefers the definition of it as "larger than one foot, two feet, and any given magnitude," in reference to linear continuous quantity. Thus, "it is greater than any finite quantity, however large." This is all well and good, and certainly resonates with Spinoza's discussions of one sort of infinity in Letter 12 or 1p15s.

However, it raises the challenge of giving a separate and independent definition of *finite*. An insight required in our mathematical interlude in Section 4 was the Frege-Dedekind idea that the intersection of all f-closed sets characterizes the finite iterates. Although Gregory's definition is hardly precise without a sharp notion of finiteness, it indicates a valuable idea. Its value is that it would explain how collections may be infinite, while one is "smaller" than the other, providing Spinoza an answer to the apparent paradoxes of infinity.

Spinoza does indeed seem to be aware of this train of thought, saying that these apparent paradoxes result "from the supposition that material substance is composed of parts" that "all those alleged absurdities (if indeed they are absurdities, which is not now under discussion)" do not derive from infinity at all, but the idea that "infinite quantity is measurable and made up of parts" (1p15s). It appears that Spinoza does not accept that they are absurdities; he seems to regard the paradoxes as based on errors. Indeed, the multitude of these parts seem central to the arguments that derive apparent contradictions.

Infinity and Paradox. There is a long history of regarding the infinite as paradoxical, roughly because we have incompatible expectations about *equally many* and *fewer than*.

Equally many: When there is a bijection between two collections, then there must be equally many members in both.

Fewer than: When one collection is a proper subset of another, then there should be fewer in the proper subset.

The idea that these two ideas are incompatible was deeply ingrained in the tradition. For instance, Galileo observes that there is a bijection between the natural numbers $1, 2, 3, 4, \ldots$ and their squares $1, 4, 9, 16, \ldots$. Thus, there must be the same number of natural numbers as of squares. However, just consider how many natural numbers do not appear in the sequence $1, 4, 9, 16, \ldots$! He regards this as a paradox that makes reasoning about the infinite treacherous; we must avoid regarding infinities as quantities.

> And in final conclusion, the attributes of equal, greater, and less have no place in infinite, but only in bounded quantities. [11, p. 41]

Nachtomy [19] reports Galileo's effect on Leibniz in the 1670s. Indeed, Cantor, in the 1870s, still had to struggle against this persistent suspicion about regarding infinite values as quantities [5].

From this point of view, the key idea of Dedekind's definition is that these ideas—*fewer than* and *equally many*, as defined above—are incompatible if and only if the collections are finite.

Indeed, for infinite collections, "fewer than" is a highly misleading term; one should instead say *no more*, i.e. there can be no more members in the proper subset than there are in the whole collection.

Ariew's citations show that Gregory of Rimini, John of Bassols', and Crescas have cogent ways of understanding the infinite that appear to have been available to Spinoza, and to avoid Galileo's paradox.

Infinite attributes. There is a persistent and vexed question about 1d6, which defines God as an "absolutely infinite being; that is, substance consisting of infinite attributes, each of which expresses eternal and infinite essence." Namely, when Spinoza speaks of infinite attributes, does he mean *infinitely many*—all but two of which are unknown to us—or does he simply mean *all* of them, without exception, even if there are only two? The latter becomes more attractive because we are told, for instance in 2p47, that we have an adequate idea of God, and if we experience only $2/\infty$, that hardly seems adequate.

Ariew, however, lines up on the side of infinitely many attributes. He stresses that for many philosophers—Aristotle among them—the All is not infinite; the heavens, for instance, could not be larger, but they are not infinite. He notes that Gregory of Rimini makes this point, drawing the conclusion that the categorematic infinite—the infinite as a substantial entity—may exceed all. On the other hand, the syncategorematic infinite, that which exceeds one, two, three, and so on, may be less than all. Galileo's squares $1, 4, 9, \ldots$ form a syncategorematic infinite, but they are not all numbers.

Thus, although Ariew concludes that "absolutely infinite" should entail "all without exception," the converse, however, he rejects as implausible for one aware of the scholastic traditions that Spinoza appears to have received.

5.2 Melamed

In a remarkable sequence of articles, Yitzhak Melamed has explored a variety of aspects of Spinoza's thought. Two that intersect with our themes are about the "Exact Science of Nonbeings" [16], which we will discuss in this section, and "Why Spinoza is not an Eleatic Monist" [18], to which we will return in Section 7. A third, "Omnis Determinatio Est Negatio" [17] brings these themes into relation to Hegel; we will postpone it for some future paper.

The Exact Science of Nonbeings. Melamed aims to develop Spinoza's views on mathematics to explain a conflict we have described above, namely the extent to which mathematics is rational and certain, while having a central dependence on imagination and its confusions or abstractions [16]. He points out some key strengths of mathematical knowledge, namely its freedom from teleology, its certainty[6], its necessity[7], and its clarity. He also attributes to Spinoza the idea that mathematics is analytic in nature, because of the role of real definitions in mathematical reasoning. This claim seems, however, anachronistic, partly because Spinoza's real definitions seem so distant from the linguistic, stipulative definitions that the strong advocates of analyticity adhered to (e.g. Carnap [4]).

By contrast, time, number, and measure are "products of the imagination's attempt to arrange the modes in a certain order, though an order which does not correctly reflect the order of modes within substance" [16, p. 9], where the order of the modes within substance apparently refers to the causal relations among the modes. They are "deceptive ways to conceive modes" and "distorted."

Melamed describes how numbers may be introduced in the mind by abstraction from our (already abstracted) concepts of collections of objects that fall under different universals. This further abstraction focuses on the equinumerous collections, and the operations that apply to them uniformly. He adds that large numbers and the mathematical laws that govern numbers in general require us to affirm the creative power of mathematical reasoning.

He disagrees with Gueroult (see Section 5.3 below) that the "terrible fall of number... opens an abyss between arithmetic and geometry" [12, pp. 201–2], and identifies a sequence of uniformities between arithmetic and geometry. This matches our view in Section 3, which we will further develop in Section 7.

He rehearses Spinoza's response to Descartes's view of mathematics, finding that Spinoza shared Descartes's respect for its clarity and certainty. Nevertheless, he showed a good deal of skepticism towards the Cartesian view that understanding nature means deducing its behavior directly from geometrical properties of extension. We noted a number of such barbs already above.

Melamed finds the solution to his puzzle in the fact that Spinoza, despite his respect for the rigor of mathematics, nevertheless regards it as a science of non-beings. Abstraction, in severing mathematical notions from their "objects in nature," makes it possible to reason uniformly with them.

Indeed, since numbers and figures are not entities that necessarily exist in nature, the knowing them does not actualize the intellect. We read

> The finite intellect in act or the infinite intellect in act must comprehend the attributes of God and the affections of God, and nothing else. (1p30)

Thus, while we might impute the rigor and certainty of mathematics "not to the defect but to the strength of [the mind's] own nature," and indeed to the freedom of the "faculty of imagining," in the terms of 2p17s, it is not a form of knowing nature or God.

[6] He quotes the *Theological-Political Treatise* that mathematical certainty is "the certainty that necessarily derives from the apprehension of what is apprehended or seen," definitely a seal of approval from any rationalist.

[7] generally by reference to the Euclidean theorem that the angles of a triangle sum to two right angles; hardly without irony now that we live in a non-Euclidean universe.

Melamed's conclusion seems persuasive: the rigor and beauty of mathematics are achieved at the cost of losing the causal efflux of modes one from another (1p28), and in fact also losing the substantial root of the modes in nature itself.

But now we are different ground. What is the nature and reason for this causal efflux of modes? Why does the root of substance issue in the tree of the modes in nature? To this question, we will return in Section 7.

5.3 Gueroult

Gueroult's beautiful and absorbing appendix on Letter 12 [12] was written decades before Ariew and Melamed's articles. It was well-known to both of them; they cited it and respectfully criticized it. We turn to it now for its treatment of the relationship between substance and the modes.

Gueroult begins from the three distinctions we rehearsed above on p. 5. Since each of these distinctions introduces two opposed notions, Gueroult aims to explain six cases of infinity.[8] Gueroult points out that "that which must be infinite by its very nature or by virtue of its definition" is *substance*. Hence, "that which is unlimited not by virtue of its essence but by virtue of its cause" is instead *mode*. "Mode is not infinite by reason of its essence, since its essence does not necessarily envelop its existence" (p. 184).[9] He points out that it is nevertheless infinite by reason of its cause, which is substance. Hence,

> every mode, in relation to its divine cause, must be conceived as 'without limits' or as infinite, at least as to the internal force which affirms it. (p. 185)

However, the mode has a merely indefinite tendency to persist; "No thing can be destroyed except by an external cause" (3p4). This provides a simple metaphysical explanation of the conatus doctrine; Gueroult cites 3p7–8:

> The conatus with which each thing endeavors to persist in its own being is nothing but the actual essence of the thing itself. (3p7)
>
> The conatus with which each single thing endeavors to persist in its own being does not involve finite time, but indefinite time. (3p8)

The conatus appears as a form of the infinity of substance—the essence of which is to exist—incorporated into each mode as the internal source of its persistence.

A similar point holds for the infinite modes (p. 188), which Gueroult exhibits as "the thing infinite insofar as without limits." Indeed, he writes, "for it is not *its own nature*, but *the nature of substance* that excludes limitation from it" (emphasis in original); Gueroult cites the second part of the demonstration of the infinite modes, i.e. 1p21d, to explain this. Substance is present indivisibly and

[8] Ariew finds this *outré*, commenting that there are in fact only three meaningful combinations of the properties [2, note 16]. However, Gueroult's decomposition into the six different aspects has value, even though we will not discuss all of them.

[9] "Envelop" occurs frequently in this translation; one supposes it corresponds to Shirley's "involve," which certainly reads better in English.

everywhere in extension, or equivalently in thought: "its infinity, as immensity, is, just like all other kinds of infinity, an absolute internal affirmation of its existence" (p. 189).

Thus, substance is seen as providing an infinity, an *absolute affirmation*, at the root of the infinite modes, as of the finite modes. With this we have now documented the view of infinity as positivity that we suggested in our introduction.

After focusing on positivity as a logical notion in Section 6, we will again turn to Gueroult for the backbone of Section 7.

6 An Interlude: Logic of positivity

Positivity would seem to be a logical notion as well as a metaphysical one. Its logical significance turns out to provide a startling interpretation of Spinoza's insistence on the uniqueness of substance.

Positive operators and strictly geometric theories. A logical operator is positive iff, when changing the truth value of an argument to which it is applied from false to true can never change the truth value of the result from true to false. For instance, *conjunction* $\wedge$ is positive because changing the truth value of ψ from false to true can never change the truth value of $\phi \wedge \psi$ from true to false. It may not be enough to make it true, in case ϕ is false, but it can never do harm. Similarly, *disjunction* $\vee$ is positive.

By contrast, *implication* $\Rightarrow$ is not positive. Changing the truth value of the hypothesis from false to true can change the truth value of an implication from true to false. For instance, *if* $2 > 3$*, then pigs can fly* is true (as a material conditional). However, if we make the hypothesis true, by substituting $2 < 3$ for $2 > 3$, then the formula becomes false, because the pigs remain earthbound. Naturally, negation $\neg$ is not positive.

Of the quantifiers *for all* $\forall$ and *there exists* $\exists$, we also count *there exists* as positive. It is preserved as we enlarge the domain of discourse. As soon as there exists an x satisfying $\phi(x)$, adding more objects to the domain may provide more values satisfying $\phi(x)$, but it can never destroy the one we have. The universal quantifier *for all* is the opposite. Adding to the domain may falsify $\forall x \,.\, \phi(x)$, so the new values may not satisfy $\phi(x)$.

Formulas built up using only positive operators $\wedge, \vee$, and $\exists$ are called *positive formulas*. They have the property that they are preserved under *homomorphisms*, which are maps from one structure to another that preserve the truth of true atomic formulas. By contrast, formulas involving $\neg, \Rightarrow$, and $\forall$ are not preserved under all homomorphisms. In our case, we will speak of *strictly positive formulas*, since we will not include a logical constant of *falsehood*.[10]

[10] *Falsehood* is (oddly) positive, since it remains constantly false; its alteration can never change the truth value of any larger formula, since it can never become true. We use the word *strictly* to signal that we will not include the constant *Falsehood* among our logical constants, as would be appropriate in other contexts. See [8] for a comprehensive study of geometric

A *strictly geometric theory* is a set of axioms that are closed formulas of the form:

$$\forall x_1, \ldots, x_n \,.\, \phi \Rightarrow \psi$$

where ϕ, ψ are strictly positive formulas. That is, *for all* and *implication* may appear, but only at the "very top" of the formulas. "Lower down," meaning inside ϕ and ψ, they do not appear. The models of strictly geometric theories—because the conditions within hypotheses ϕ and conclusions ψ are purely positive—have are closed under two operations. We allow the case in which the variables $x_1, \ldots, x_n$ are vacuous, and we also allow the case in which the hypothesis ϕ is the constantly true formula *true*.

Models of strictly geometric theories. We will adduce three properties of the set of models $\mathfrak{M}(T)$ of a strictly geometric theory T. We will write $\mathcal{A} \in \mathfrak{M}(T)$ or $\mathcal{A} \models T$ if $\mathcal{S}$ is a model of

First, every strict geometric theory T has models; $\mathfrak{M}(T) \neq \emptyset$. This follows from an algorithm called *the chase* due to Fagin, Kolaitis, and Popa [9]. Indeed, since any set of closed atomic formulas (possibly using additional individual constants) is also a strict geometric theory T', then by applying the previous claim to $T \cup T'$, we can build a model in which specific values have the additional desired properties mentioned in T'.

Second, we can always "add equations" to a given model. Suppose that $\mathcal{A} \in \mathfrak{M}(T)$ is a model of a strictly geometric theory T, and the domain of $\mathcal{A}$ is A. Let $\{\langle a_i, b_i \rangle\}_{i \in I} \subseteq A \times A$ be a set pairs of values in the domain. Then there is a model $\mathcal{B}$ of T and a homomorphism $h_0 \colon \mathcal{A} \to \mathcal{B}$ such that:

1. h_0 equates each pair: $h_0(a_i) = h_0(b_i)$ for each $i \in I$; and
2. h_0 is the least among such homomorphisms. Namely, if $\mathcal{C}$ is a model of T and $h_1 \colon \mathcal{A} \to \mathcal{C}$, and $h_1(a_i) = h_1(b_i)$ for each $i \in I$, then there exists a homomorphism $k \colon \mathcal{B} \to \mathcal{C}$ such that $h_1 = k \circ h_0$.

Property 2 says that h_0 does no more work than any h_1 that identifies those pairs. Since each homomorphism adds information about equalities between values and additional properties of the values, If $h_1 = k \circ h_0$, then h_1 adds at least as much information as h_0. So this is a kind of minimality.

Thus, we can always choose to identify elements of the domain of a model together, and there is a model compatible with those choices, and indeed among all such models, one that identifies elements only as necessary, i.e. minimally.

Third, suppose that we have two T-models $\mathcal{A}, \mathcal{B}$, and an injective function $f \colon A \to B$ between their domains. Then f allows us to superimpose the content of $\mathcal{A}$ on top of $\mathcal{B}$, obtaining a new model $\mathcal{C}$ with the same domain B as $\mathcal{B}$. We stipulate that for an atomic formula ϕ with the variables $x_1, \ldots, x_n$, and for a sequence of values $b_1, \ldots, b_n \in B$, that $\mathcal{C} \models_\eta \phi$ under the variable assignment η that maps each x_i to b_i, iff either:

logic.

1. $\mathcal{C} \models_\eta \phi$ or else

2. there exist $a_1, \ldots, a_n \in A$ such that $\mathcal{A} \models_\theta \phi$, where θ is the variable assignment that maps each x_i to a_i, and $f(a_i) = b_i$ for each i.

The form of the strictly geometric axioms now ensures that each axiom is satisfied in $\mathcal{C}$. If an instance of a hypothesis is satisfied in $\mathcal{B}$, then a corresponding instance of the conclusion holds in $\mathcal{C}$ because it already held in $\mathcal{B}$. If an instance of a hypothesis is satisfied in $\mathcal{C}$ because some f-preimage of it is satisfied in $\mathcal{A}$, then a corresponding f-preimage of an instance of the conclusion also holds in $\mathcal{A}$. The second condition allows us to transport the instance of the conclusion to $\mathcal{C}$ using f.

Observe now that f may be regarded as a homomorphism $f : \mathcal{A} \to \mathcal{C}$, while the identity function id_B is a homomorphism $\mathsf{id}_B : \mathcal{B} \to \mathcal{C}$.

Thus, we can combine the content of any two models if we have an injective function f from the domain of one to the domain of the other. We obtain a sort of least upper bound $\mathcal{C}$ for $\mathcal{A}, \mathcal{B}$ under the function f.

We can extend this property to the case where we have a family of pairs $\{\langle \mathcal{A}_i, f_i \rangle\}_{i \in I}$ where, for a single $\mathcal{B}$, $f_i : A_i \to B$. Then we can build a sort of least upper bound $\mathcal{C}$ for $\mathcal{B}$ and all of the $\mathcal{A}_i$.

Strictly geometric theories have final models of cardinality 1. From the three properties just given, it follows that each strictly geometric theory T has a so-called *final model* $\mathcal{F}$:

Theorem 1 *Every strictly geometric theory T has a T-model $\mathcal{F}$ such that:*

1. *the domain F of $\mathcal{F}$ contains exactly one element $F = \{a_0\}$;*

2. *for every T-model $\mathcal{A}$, there is a unique homomorphism $h_\mathcal{A} : \mathcal{A} \to \mathcal{F}$;*

3. *for each atomic formula ϕ with the variables $x_1, \ldots, x_n$, let η be the variable assignment that assigns a_0 to each variable x_i; then $\mathcal{F} \models_\eta \phi$.*

This theorem explains the logical content of *positivity*: When we have a purely positive set of laws that form a strictly geometric theory T, then those laws admit a most informative model $\mathcal{F}$. Every atomic formula is satisfied in $\mathcal{F}$. Moreover, that model has a domain with a unique element. Every other model is essentially a partial view of $\mathcal{F}$; the homomorphism $h_\mathcal{A} : \mathcal{A} \to \mathcal{F}$ explains how $\mathcal{A}$ may be regarded as an approximation to $\mathcal{F}$.

Thus, a final model has the core characteristics of Spinoza's substance. In particular, it consists of a unique entity (item 1); it incorporates all partial views (item 2); and it encompasses all truths (item 3).

One finds it startling that the idea of infinity as positivity leads to a logical theory that matches the metaphysical constraints so nicely. However, the real action is to relate substance with its modes.

7 Substance and its modes: Nature

We have already seen from Gueroult that the finite modes are nevertheless infinite by virtue of their cause, although not by their essence, which does not involve existence. Thus, they are finite and contingent; they can just as well not exist as exist. But then, why do they exist?

Why diversity exists. Melamed turns to this challenge in an article, "Why Spinoza is not an Eleatic Monist (Or Why Diversity Exists)" [18]. The problem here is simply stated. Substance is conceived through itself (1d3), and "is by nature prior to its affections," i.e. the modes (1p1), which are however conceived only through substance (1d5). Thus substance is independent, and the modes are asymmetrically dependent upon it. So presumably substance could exist without the modes? But in that case, why do the modes ("diversity") exist?

If we do not provide a rationally motivated answer, we would have to conclude that they *cannot* exist: The Principle of Sufficient Reason would certainly demand that the existence of the natural world has an explanation.

A theory popular among the German idealists provides one possible reply; Melamed cites Salomon Maimon as a predecessor to Hegel as holding to this approach. It was the idea that Spinoza—formerly reviled as an atheist—was the very opposite. Rather than denying the existence of God, he asserted the existence of God, instead denying that anything *other* than God exists. Since the latter would be the cosmos, Spinoza would be not an atheist but an *acosmist*. Hegel's statement of this idea is found in the *History of Philosophy* [13, vol. 3, p. 281]. Melamed enumerates a litany of reasons why this claim, interesting as it is, seems untrue to Spinoza.

Melamed proposes an extremely interesting alternative. He notes 1p34, "God's power is his very essence," and the words of 2p3 that "it is as impossible for us to conceive that God does not act as it is to conceive that he does not exist." These suggest the idea that God as *natura naturans* must be active. In the absence of *natura naturata*, *natura naturans* could act only on itself. Thus, it "would be just as active as it is passive." [18, 213–14] The essentially causal character of existence, its active character, expresses itself only through giving rise to *natura naturata*.

In this regard, God's absolute infinity, which may be identical with his uniqueness, distinguishes him from the *natura naturata* which contains the modes. The latter is at most infinite in its own kind, and is distinguished from *natura naturans* by its divisibility, passivity, dependence.

Existence and persistence. Although any particular mode is finite and contingent—it can just as well not exist as exist—each mode nevertheless does have a tendency to remain in existence. This shows itself in its destruction always being due to an external cause (3p4): "the definition of any thing affirms, and does not negate, the thing's essence" (3p4d). Indeed, a thing is "of a contrary nature" to anything that can destroy it (3p5), and thus each thing "endeavors to persist in its own being" (3p6). The demonstration of this last proposition reminds us:

> Particular things are modes whereby the attributes of God are expressed in a definite and determinate way (1p25c), that is (1p34), they are things which express in a definite and determinate way the power of God whereby he is and acts.... (3p6d)

This passage seems a direct confirmation of Melamed's hypothesis that God's power or activity requires something determinate and finite (1d2) for its expression, and thus something defined by negation or limitation

However, although each finite mode could exist or not, compatibly with its essence, when they do exist, they have a tendency to continue in existence. This property of a finite mode—its endeavor to persist—receives the name *conatus* in 3p7–8. This conatus is "the actual essence of the thing itself" (3p7). The following proposition immediately connects this with finitude:

> The conatus with which each single thing endeavors to persist in its own being does not involve finite time, but indefinite time. (3p8)

From this "internal" point of view, "the infinity of the internal force is thus resolved into" its conatus, i.e. "a simple *indefinite tendency* to exist and to persevere in being" [12, p. 185]; Gueroult compares this to the Cartesian indefinite, except that it is an objective absence of limits rather than an absence of subjective limits known by us; it is indefinite *an sich* rather than *für uns*.

A similar but more powerful exclusion of limits applies to the infinite modes. Here Gueroult divides his discussion ("*VI. Case no. 3: The thing infinite insofar as without limits*" [12, p. 187ff.]) into two portions, depending as the infinite mode is conceived by the imagination or by reason.

7.1 Geometry: The mathematics of extension

One suspects that the infinite mode that follows from the attribute of extension (cp. 1p21d, last sentence), when conceived by the imagination, is in fact *geometric space*, i.e. what we would call Euclidean space. The nature of imagination is in fact confusion, as manifested in the transcendentals and universals (2p40s1), and also as manifested in number (Letter 50 to Jelles). With number, the confusion—more politely called abstraction—is highly structured, namely collections may be confused arithmetically with each other when there is a bijection between them, i.e. they are equinumerous.

With geometry, there is also great structure to the confusion that characterizes Euclidean space. In particular, we can think of this confusion as given by a set of *transformations* that map points of space to other points of space—and thus figures (sets of points) to other figures—such that the geometric properties do not distinguish between any figure and its image under these transformations. In fact, Euclidean geometry is characterized by a set of transformations. They are the ones that are built by composing the following basic kinds of transformations:

Translations: A translation is a “sliding” transformation that moves every point the same amount and in the same direction;

Rotations: A rotation leaves a point, the center, unchanged; every other point moves to a new position the same distance from the center and a fixed angle from its original position.

Scaling: A scaling transformation leaves a point, the center, unchanged; every other point moves closer to the center by a fixed proportion or farther from it by a fixed proportion, remaining on the same line from the center.

Reflections: A reflection flips a left hand into a right hand, leaving distances unchanged; or a clockwise-tightening screw into a clockwise-loosening one.

These transformation carry every figure to a *similar* figure, or provide a way of defining similarity. They characterize Euclidean geometry, as different choices of transformations characterize other geometries. For instance, if we omit the scaling transformations, we obtain a geometry with an intrinsic unit of measure that always stays the same. Thus, geometry and its transformations provide a highly structured approach to confusing extended objects.

However, the geometric transformations are indifferent to causality, i.e. they do not preserve causal relations. Causality is the heart of reason, of the understanding. Melamed [16, pp. 15–16] attributes to Spinoza a Leibniz-like view of the non-reality of space based on the Principle of Sufficient Reason: The preservation of all geometric properties under these transformations (for instance, the translations) means that no spatial regions can be preferable to any other, similarly shaped, spatial region, for instance for my left hand to occupy now. It is in this sense that spatial regions are nonbeings, according to Melamed.

Despite not characterizing any individual, these generalities are nevertheless a deep part of our knowledge:

> That which is common to all things (see Lemma 2 above) and is equally in the part as in the whole, does not constitute the essence of any one particular thing. (2p37)
>
> Those things that are common to all things and are equally in the part as in the whole, can be conceived only adequately. (2p38)

Scientific knowledge, i.e. universal claims deductively grounded—whether viewed Cartesian or Aristotelian—would thus sacrifice its ability to speak of the essences of individual things to the need to capture what is common. Spinoza recognizes the epistemological advantage of generality in 2p38.

7.2 Causal reasoning

Causal reasoning is how we understand nature. The infinite modes, such as the face of the whole universe (Gueroult, p. 188), is a whole in which each finite mode is linked with the cause of its arising, the cause of its passing away, and the causes of its successive changes. The infinity of this universe is ensured by its role as manifesting the affections of substance. Were it "limited, part of substance would be deprived of affections, and, so, annihilated," Gueroult tells us, relying on 1p21d. The early portion of Part 3 reinforces this point.

The causal nexus of the modes binds them together while ensuring that each remains distinguished by its position in the efflux. As a consequence, *causal* understanding can be a knowledge of *beings* rather than non-beings. Their individuality and distinct explanations can be acknowledged, meaning that this form of knowing is compatible with the Principle of Sufficient Reason.

7.3 Divisibility

Gueroult also stresses that the infinite divisibility of the modes is deeply related to the indivisibility of substance.

> Substance is thus, with regard to its nature, equally, that is, entirely, in the totality of its modes as it is in each of them, in each of them as it is in each of them as it is in each of their parts, and in each of their parts as in each of the parts of these parts, etc., to infinity. [12, p. 195]

Thus, in the case of an attribute, e.g. extension, "the *nature* of extension remains complete, that is, identically what it is, in the least of its particles." Hence, each mode of extension contains a continuum within it, constituting a world of nested modes. This seems an almost Leibnizian thought; we may compare a vivid sentence of Leibniz's in a letter to Johann Bernoulli (18 Nov. 1698) :

> ...there could be, indeed, there have to be, worlds not inferior in beauty and variety to ours in the smallest motes of dust, indeed, in tiny atoms. [15, p. 169]

The presence of substance within each mode also ensures the know-ability of the whole through any of its parts:

> The idea of any body or particular thing existing in actuality necessarily involves the eternal and infinite essence of God. (2p45)

Spinoza immediately concludes that this knowledge of the essence of God is adequate and perfect (2p46). Nature, we might thus say, is holographic: Every mode within it suffices to allow us to understand "the eternal and infinite essence of God" in perfect adequacy.

8 A Glance Forward to Leibniz

There are differences, both obvious and profound, between Spinoza and Leibniz. Among the obvious ones are their opinions about organized religion, and their political tendencies.

More central for first philosophy is God's will. Leibniz believed strongly that God's activity should be understood as exercising a faculty of willing, and thus that the nature of the world shows an optimum—chosen by God—among an infinite collection of possibilities, each of them extremely rich in detail (cf. e.g. *Monadology*, §§53–55, [15, p. 220]). Spinoza, however, stated that "God does not act from freedom of will" (1p32c1), and saw the nature of substance and the efflux of its modes as proceeding with necessity. Certainly God (or Nature) does not proceed with an end in view (Appendix to Part 1).

An even sharper contrast concerns the locus of the law that governs causality or development. Leibniz held the remarkable "Predicate in Notion" thesis, namely that the individual concept of any substance contains the law of its development, from which all details about it would follow (cf. e.g. *Discourse on Metaphysics*, §13, [15, p. 45], in which Caesar crosses the Rubicon). It might, however, require an infinite deduction to extract the conclusions from this infinitely rich premise. By contrast, Spinoza believed that the progression of finite modes is determined by the other finite modes of the same attribute (1p28). Thus, for particular things, Leibniz held that they are essentially independent of other particular things—though utterly in harmony with them—while Spinoza saw them as causally dependent on each other. This causal indissolubility welds the finite modes into a rationally understandable nature.

That said, there are also deep affinities on this point. For instance, Spinoza's attribution of conatus to "each single thing" certainly foreshadows Leibniz's view of the appetition (*Monadology*, §15, [15, p. 215]) that characterizes the infinity of monads associated with all the parts of extended matter. We have also seen that each of them insists on the infinite divisibility—indeed, infinite actual division—of extended reality.

Even though the Predicate in Notion principle is distinctively Leibnizian, there is a kind of analogy with Spinoza's view of the causal necessity of the occurrences in nature. Indeed, we can regard the Predicate in Notion principle as a kind of justification for regarding monads as substances. After all, it implies that each monad is evolving according to a law peculiar to itself. This supplies the independence that the notion of substance requires. In Spinoza, of course, it is only the whole of nature that can amount to substance. However, it does so because it evolves according to a law peculiar to itself. That is why:

> Nothing in nature is contingent, but all things are from the necessity of the divine nature determined to exist and to act in a definite way. (1p29)

Thus, nature as a whole proceeds according to a law peculiar to the substance of which it consists. Thus, there is a sense in which Leibniz's monads represent a transposition of substance—in a sense understandable to Spinoza—from the totality of nature to each individual thing. This transposition and multiplication is possible only because Leibniz can preserve the totality by ensuring that each of the monads mirrors the whole, even if from its own point of view. Thus, in Leibniz we find a sort of replication everywhere of a notion of substance that is singular in Spinoza.

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