An Application of the Regularity Lemma in Generalized Ramsey Theory

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Abstract: Given graphs G and H, an edge coloring of G is called an (H,q)-coloring if the edges of every copy of $H \subset G$ together receive at least q colors. Let $r(G,H,q)$ denote the minimum number of colors in a (H,q)-coloring of G. In [9] Erdős and Gyárfás studied $r(K_n,K_p,q)$ if p and q are fixed and n tends to infinity. They determined for every fixed p the smallest q (denoted by q_{lin}) for which $r(K_n,K_p,q)$ is linear in n and the smallest q (denoted by q_{quad}) for which $r(K_n,K_p,q)$ is quadratic in n. They raised the problem of determining the smallest q for which we have $r(K_n,K_p,q) = \binom{n}{2} - O(n^2)$. In this paper by using the Regularity Lemma we show that if $q > q_{quad} + \lceil \frac{\log_2 p}{2} \rceil$, then we have $r(K_n,K_{p},q) = \binom{n}{2} - O(n^2)$.
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1. INTRODUCTION

1.1. Notation and Definitions

For basic graph concepts see the monograph of Bollobás [3]. $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of the graph G. K_n is the complete graph on n vertices, and $K_{n,n}$ is the complete bipartite graph between two n-sets. (A,B) or (A,B,E) denote a bipartite graph $G=(V,E)$, where $V=A\cup B$, and $E\subset A\times B$. In general, given any graph G and two disjoint subsets A,B of $V(G)$, the pair (A,B) is the graph restricted to $A\times B$. $N(v)$ is the set of neighbors of $v\in V$. Hence the size of $N(v)$ is $|N(v)|=deg(v)=deg_G(v)$, the degree of v. For a vertex $v\in V$ and set $U\subset V-\{v\}$, we write $deg(v,U)$ for the number of edges from v to U. We denote by $e(A,B)$ the number of edges of G with one endpoint in A and the other in B. For non-empty A and B,

$$d(A,B)=\frac{e(A,B)}{|A||B|}$$

is the *density* of the graph between A and B.

Definition 1.1. *The pair (A,B) is ε-***regular** *if*

$$X\subset A,\ Y\subset B,\ |X|>\varepsilon|A|,\ |Y|>\varepsilon|B|$$

imply

$$|d(X,Y)-d(A,B)|<\varepsilon,$$

otherwise it is ε-irregular.

A hypergraph $\mathcal{F}$ is called $k-$*uniform* if $|F|=k$ for every edge $F\in\mathcal{F}$. A k-uniform hypergraph $\mathcal{F}$ on the set X is $k-$*partite* if there exists a partition $X=X_1\cup\cdots\cup X_k$ with $|F\cap X_i|=1$ for every edge $F\in\mathcal{F}$ and $1\le i\le k$.

1.2. Generalized Ramsey Theory

In the classical multicoloring Ramsey problem, we are looking for the minimum number n such that every k-coloring of the edges of K_n yields a monochromatic K_p. For each n below this threshold, there is a k-coloring such that every K_p is colored with at least 2 colors. A far-reaching generalization of this concept leads to the following definition: given graphs G and H, and an integer $q\le|E(H)|$, an $(H,q)-$*coloring* of G is a coloring of $E(G)$ in which the edges of every copy of $H\subset G$ together receive at least q colors. Let $r(G,H,q)$ denote the minimum number of colors in an (H,q)-coloring of G. Thus for example determining $r(K_n,K_p,2)$ exactly is hopeless, since it is equivalent to determining the classical Ramsey numbers for multicolorings. The study of $r(G,H,q)$ has received significant attention lately (see [1,2,6,9,13,14,16,17). It was first studied in this form by Elekes, Erdős and Füredi for the special case $r(K_n, K_p, q)$ (as described in Section 9 of [7]). Then Erdős and Gyárfás 15 years later returned to the problem in [9]. Among many other interesting results and problems, in [9] using the Local Lemma they gave the general upper bound

$$r(K_n, K_p, q) \leq c_{p,q} n^{\frac{p-2}{\binom{p}{2}-q+1}}. \tag{1}$$

Furthermore, for every p they determined q_{lin}, the smallest q for which $r(K_n, K_p, q)$ is linear in n, ($q_{lin} = \binom{p}{2} - p + 3$) and q_{quad}, the smallest q for which $r(K_n, K_p, q)$ is quadratic in n, ($q_{quad} = \binom{p}{2} - \lfloor \frac{p}{2} \rfloor + 2$).

They raised the striking question if q_{lin} is the only q value which results in a linear $r(K_n, K_p, q)$. In the direction of this question, in [16] we studied the behavior of $r(K_n, K_p, q)$ between the linear and quadratic orders of magnitude, so for $q_{lin} \leq q \leq q_{quad}$.

In particular we showed that we can have at most log p values of q which give a linear $r(K_n, K_p, q)$. The first interesting case is $p = 5$ for which $q_{lin} = 8$. What is the growth rate of $r(K_n, K_5, 9)$? In [1] it is shown by using a construction of Behrend for a set of integers with no 3-term arithmetic progressions that

$$\frac{1+\sqrt{5}}{2} n - 3 \leq r(K_n, K_5, 9) \leq 2n^{1+\frac{c}{\sqrt{\log n}}}.$$

See also [14] for a related article. Another interesting special case is $r(K_n, K_4, 3)$. In [13] (see also [6])) it is shown that

$$r(K_n, K_4, 3) \leq e^{o(\log n)}.$$

Another interesting problem raised in [9] was to determine the smallest q for which $r(K_n, K_p, q) = \binom{n}{2} - o(n^2)$. They showed that for q_{quad} this is not true, hence this value of q has to be larger than q_{quad}. They remarked that determining this transition seems to be a difficult problem. This problem is related to a well-known problem of Brown, Erdős and T. Sós on hypergraphs ([4]): determine the minimum number of edges of an r-uniform hypergraph on n vertices which guarantees that there exist k vertices spanning at least s edges. Special cases of this problem appeared in [5,8]. The problem whether the minimum is $o(n^2)$ for $r = 3$, $k = 6$, $s = 3$ was asked in [4] and answered affirmatively by Ruzsa and Szemerédi in [15].

Using the connection with the Ruzsa-Szemerédi theorem, in [9] for $p = 9$ it was shown that $r(K_n, K_9, \binom{9}{2} - 2) = \binom{n}{2} - o(n^2)$. For general p they showed

$$r\left(K_n, K_p, \binom{p}{2} - \left\lfloor \frac{p-4}{3} \right\rfloor\right) = \binom{n}{2} - o(n^2).$$

In this paper we significantly improve on this result. We show by applying the Regularity Lemma of Szemerédi ([18]) that if $q > q_{quad} + \lceil \frac{\log_2 p}{2} \rceil$, then we have

$r(K_n, K_p, q) = \binom{n}{2} - o(n^2)$. Thus not long after $r(K_n, K_p, q)$ becomes quadratic, it jumps up to $\binom{n}{2} - o(n^2)$. More precisely our main result is the following.

Theorem 1.1. *For any integer $p \geq 3$, if $q > q_{quad} + \lceil \frac{log_2 p}{2} \rceil$, then*

$$r(K_n, K_p, q) = \binom{n}{2} - o(n^2).$$

The general definition $r(G, H, q)$ given above is introduced in [2] by Axenovich, Füredi, and Mubayi. Among other results, in [2] by generalizing the upper bound (1), they showed that if $|V(G)| = n$, $|V(H)| = v$, $|E(H)| = e$ and $1 \leq q \leq e$, then there is a constant $c = c(H, q)$ such that

$$r(G, H, q) \leq cn^{\frac{v-2}{e-q+1}}. \tag{2}$$

Furthermore, for the bipartite case $r(K_{n,n}, K_{p,p}, q)$, they determined for every p the smallest q ($q^b_{lin} = p^2 - 2p + 3$) for which $r(K_{n,n}, K_{p,p}, q)$ is linear in n, and the smallest q ($q^b_{quad} = p^2 - p + 2$) for which $r(K_{n,n}, K_{p,p}, q)$ is quadratic in n.

In [17] we studied the behavior of the function $r(K_{n,n}, K_{p,p}, q)$ between the linear and quadratic orders of magnitude, so for $q^b_{lin} \leq q \leq q^b_{quad}$. We showed that we can have at most $\log p + 1$ values of q which give a linear $r(K_{n,n}, K_{p,p}, q)$.

From the above survey it is clear that several interesting open problems remain in the area.

In the next section we provide the tools including the Regularity Lemma. Then in Section 3 we apply the Regularity Lemma to obtain our main lemma. Finally in Section (Proof ot Theorem 1.1) we prove the theorem.

2. TOOLS

In the proof of Theorem 1.1 the Regularity Lemma of Szemerédi ([18]) plays a central role. Here we will use the following variation of the lemma.

Lemma 2.1 (Regularity Lemma – Degree form). *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex-set V into $l + 1$ sets (so-called clusters) $C_0, C_1, \ldots, C_l$, and there is a subgraph $G' = (V, E')$ with the following properties:*

- $l \leq M$,
- $|C_0| \leq \varepsilon |V|$,
- *all clusters C_i, $i \geq 1$, are of the same size,*
- $deg_{G'}(v) > deg_G(v) - (d + \varepsilon)|V|$ *for all* $v \in V$,
- $G'|_{C_i} = \emptyset$ (*C_i are independent in G'*),
- *all pairs $G'|_{C_i \times C_j}$, $1 \leq i < j \leq l$, are ε-regular, each with a density 0 or exceeding d.*

This form (see [12]) can easily be obtained by applying the original Regularity Lemma (with a smaller value of ε), adding to the exceptional set C_0 all clusters incident to many irregular pairs, and then deleting all edges between any other clusters where the edges either do not form a regular pair or they do but with a density at most d.

We will also use a simple but useful result of Erdős and Kleitman ([10], see also on page 1300 in [11]).

Lemma 2.2. *Every k-uniform hypergraph $\mathcal{F}$ contains a k-partite k-uniform hypergraph $\mathcal{H}$ with*

$$\frac{|\mathcal{H}|}{|\mathcal{F}|} \geq \frac{k!}{k^k}.$$

For technical reasons we define the following two strictly decreasing sequences a_i and b_j of positive integers. We start with $a_0 = \lfloor \frac{p}{2} \rfloor - \lceil \frac{\log_2 p}{2} \rceil$ and $a_1 = \lfloor \frac{a_0}{2} \rfloor$. Roughly speaking $a_{i+1} = \lfloor \frac{a_i}{2} \rfloor$ but for every second odd a_i we have to add 1.

The two sequences are defined recursively. Assuming $a_0, a_1, \ldots, a_i$ are already defined, the sequence $b_1, b_2, \ldots, \ldots, b_{i'}$ is just the subsequence consisting of the odd a_i-s which are greater than 1. Then we define

$$a_{i+1} = \begin{cases} \lceil \frac{a_i}{2} \rceil & \text{if } a_i = b_j \text{ for an even } j \\ \lfloor \frac{a_i}{2} \rfloor & \text{otherwise} \end{cases}$$

Furthermore if a_{i+1} is odd and greater than 1, then $b_{i'+1} = a_{i+1}$.

So, for example, if $a_0 = 2^k$, the sequence $a_i, 1 \leq i$, is just all the powers of 2 from a_0 to 1, while there are no b_j-s. As another example, if $a_0 = 22$, then $a_1 = 11$, $a_2 = 5$, $a_3 = 3$, $a_4 = 1$, $b_1 = 11$, $b_2 = 5$, and $b_3 = 3$. Let l_p be the smallest integer for which $a_{l_p} = 1$.

We will need the following simple lemma.

Lemma 2.3. *For $1 \leq i \leq l_p$, we have*

$$a_i < \frac{a_0}{2^i} + 1. \tag{3}$$

Proof. We use induction on $i = 1, 2, \ldots, l_p$. It is true for $i = 1$. Assume that it is true for i and then for $i + 1$ from the definition of a_{i+1} we get

$$a_{i+1} \leq \frac{a_i + 1}{2} < \frac{\frac{a_0}{2^i} + 1 + 1}{2} = \frac{a_0}{2^{i+1}} + 1,$$

and thus proving Lemma 2.3. ■

This lemma immediately gives the bound

$$l_p \leq \lceil \log_2 a_0 \rceil \leq \lceil \log_2 p \rceil - 1. \tag{4}$$

Let l'_p be the number of b_j-s among $a_0, \dots, a_{l_p-1}$, and let $f_p(k)$ be 1 if the cardinality of $\{a_0, \dots, a_{k-1}\} \cap \{b_1, \dots, b_{l'_p}\}$ is an even number, and 0 otherwise. Examination of the cases of the definitions of a_k and b_k reveals that $f_p(k+1) = 2a_{k+1} - a_k + f_p(k)$ for all p and $1 \le k \le l_p$. Then we have the following.

Lemma 2.4. *For any* $1 \le k \le l_p$

$$\sum_{j=1}^{k} a_j = a_0 - a_k - 1 + f_p(k).$$

Proof. The lemma is clearly true for $k = 1$, and we assume it is true for an arbitrary k. Then

$$\begin{aligned}\sum_{j=1}^{k+1} a_j &= a_0 - a_k - 1 + f_p(k) + a_{k+1} \\ &= a_0 - a_{k+1} - 1 + (2a_{k+1} - a_k + f_p(k)) \\ &= a_0 - a_{k+1} - 1 + f_p(k+1).\end{aligned}$$

■

3. APPLYING THE REGULARITY LEMMA

We will prove the following lemma by applying the Regularity Lemma.

Lemma 3.1. *For every $c_1 > 0$, $c_2 \ge 1$ there are positive constants η, n_0 with the following properties. Let G be a graph on $n \ge n_0$ vertices with $|E(G)| \ge c_1 n^2$ that is the edge disjoint union of matchings $M_1, M_2, \dots, M_m$ where $m \le c_2 n$. Then there exist an $1 \le i \le m$ and $A, B \subset V(M_i)$ such that*

- $(A \times B) \cap M_i = \emptyset$,
- $|A| = |B| \ge \eta n$,
- $\left|E(G|_{A\times B})\right| \ge \frac{c_1}{4}|A||B|$.

Proof. Let us apply the degree form of the Regularity Lemma (Lemma 1.1) with

$$d = \frac{c_1}{2} \quad \text{and} \quad \varepsilon = \frac{c_1}{6c_2}. \tag{5}$$

Let $G'' = G' \setminus C_0$. Then we have

$$\begin{aligned} deg_{G''}(v) &> deg_G(v) - (d+\varepsilon)n - |C_0| \\ &\ge deg_G(v) - (d+2\varepsilon)n \quad \text{for all} \quad v \in V(G''). \end{aligned}$$

Thus using (5)

$$|E(G'')| = \frac{1}{2}\sum_{v\in V(G'')} deg_{G''}(v) > \frac{1}{2}\sum_{v\in V(G'')} deg_G(v) - \frac{d+2\varepsilon}{2}n^2$$
$$= \frac{1}{2}\sum_{v\in V(G)} deg_G(v) - \frac{1}{2}\sum_{v\in C_0} deg_G(v) - \frac{d+2\varepsilon}{2}n^2 \geq |E(G)| - \frac{d+3\varepsilon}{2}n^2 \geq \frac{c_1}{2}n^2.$$

Hence there is an $1 \leq i \leq m$ such that

$$\left|M_i|_{G''}\right| > \frac{c_1}{2c_2}n = 3\varepsilon n. \tag{6}$$

Write $U = V(M_i|_{G''})$ for the vertex set of $M_i|_{G''}$. (6) implies that $|U| > 6\varepsilon n$. Write also $U_i = U \cap C_i$. Define $I = \{i \mid |U_i| > 3\varepsilon|C_i|\}$, and set $U' = \cup_{i\in I} U_i$ and $U'' = U\backslash U'$. Clearly $|U''| \leq 3\varepsilon n$. Since $|U| > 6\varepsilon n$, we have two vertices $u, v \in U'$ adjacent in $M_i|_{G''}$. Let $u \in C_i$ and $v \in C_j$. In G'' we have at least one edge between C_i and C_j, and hence we must have a density more than $d = \frac{c_1}{2}$ between them. Consider U_i and U_j. A is an arbitrary subset of U_i with $|A| = \lfloor\varepsilon|C_i|\rfloor + 1 > \varepsilon|C_i|$. B is an arbitrary subset of U_j with $|B| = \lfloor\varepsilon|C_j|\rfloor + 1 > \varepsilon|C_j|$ and $(A \times B) \cap M_i = \emptyset$. This is possible since

$$|U_j| > 3\varepsilon|C_j| > 2\lfloor\varepsilon|C_j|\rfloor + 2,$$

if $n \geq n_0$. Then the first property of A, B in the lemma is clearly satisfied. For the second property we can choose $\eta = \frac{\varepsilon(1-\varepsilon)}{M(\varepsilon)}$. Finally for the third property, ε-regularity of the pair (C_i, C_j) implies that the density between A and B is more than $d - \varepsilon \geq \frac{c_1}{4}$. This means

$$\left|E(G|_{A\times B})\right| \geq \frac{c_1}{4}|A||B|,$$

and thus completing the proof of the lemma. ∎

4. PROOF OF THEOREM 1.1

Let $p \geq 3$ and $q > q_{quad} + \lceil\frac{\log_2 p}{2}\rceil$.

Assume indirectly that there is a constant $c > 0$ such that

$$r(K_n, K_p, q) \leq \binom{n}{2} - cn^2.$$

From this assumption we will get a contradiction. Consider a fixed (p, q)-coloring of K_n with at most $\binom{n}{2} - cn^2$ colors. Let us assume that n is sufficiently large.

Each color class has at most $\frac{p}{2}$ edges. Indeed, otherwise we can easily find a K_p with fewer than q colors, a contradiction. From this it follows that we can find

$\frac{2c}{p}n^2$ edge pairs in K_n such that the two edges in the pair have the same color, and this color is different for different pairs. Furthermore, out of these $\geq \frac{2c}{p}n^2$ edge pairs we can have at most $\frac{p}{2}n$ pairs where the two edges in the pair are incident to each other. Indeed, otherwise by the pigeon-hole principle there must be a vertex that is incident to at least $\frac{p}{2}$ edge pairs, and then we can find again a K_p with less than q colors, a contradiction. Thus we have at least $\frac{c}{p}n^2$ edge pairs where the two edges in the pair cover 4 vertices.

Using these pairs we define a 4-uniform hypergraph $\mathcal{F}$ on the vertices of K_n. The 4 vertices covered by the pairs form the edges of $\mathcal{F}$. If an edge of $\mathcal{F}$ has multiplicity greater than 1 (it is at most 3), then we keep only one of these edges. Using the Erdős-Kleitman theorem (Lemma 2.2) we find a sub-hypergraph $\mathcal{H}$ such that $\mathcal{H}$ is 4-partite and it still contains at least

$$\frac{4!c}{4^4 3p}n^2$$

edges. Let X_1, X_2, X_3, X_4 be the vertex classes of this 4-partite hypergraph $\mathcal{H}$. Consider the 3-uniform hypergraph $\mathcal{H}^*$ which is defined by the removal of X_1 from the vertex set of $\mathcal{H}$ and from all edges of $\mathcal{H}$. Again, if an edge of $\mathcal{H}^*$ has multiplicity greater than 1 (it can be at most 3), then we keep only one edge. Then $\mathcal{H}^*$ still has at least

$$\frac{4!c}{4^4 3^2 p}n^2$$

edges.

Consider an arbitrary $v \in X_2$ and the bipartite graph G_b^v defined by v between X_3 and X_4 such that (u, w) is an edge in G_b^v if and only if (u, v, w) is a 3-edge in $\mathcal{H}^*$. The maximum degree in G_b^v is at most $\frac{p}{2}$, since again otherwise we can find a K_p with fewer than q colors. Then we can choose a matching M_v in G_b^v such that

$$|M_v| \geq \frac{2|E(G_b^v)|}{p}.$$

Define the bipartite graph $G_b = \cup_{v \in X_2} M_v$. Then

$$|E(G_b)| \geq \frac{2(4!)c}{4^4 3^2 p^3}n^2.$$

Next by applying Lemma 3.1 iteratively in G_b, we will find a sequence of matchings $M_{v_1}, \ldots, M_{v_{l_p}}$. To obtain M_{v_1} we apply Lemma 3.1 in G_b. We can choose

$$c_1 = c_1^1 = \frac{24!c}{4^4 3^2 p^2} \quad \text{and} \quad c_2 = c_2^1 = 1.$$

M_{v_1} is the M_i guaranteed in the lemma. Denote $M_{v_1} = (A_1, B_1)$ where $A_1 \subset X_3, B_1 \subset X_4$. Lemma 3.1 also guarantees that there are $A'_1, B'_1 \subset V(M_{v_1})$ such that

- $(A'_1 \times B'_1) \cap M_{v_1} = \emptyset$,
- $|A'_1| = |B'_1| \geq \eta_1 n$,
- $|E(G_b|_{A'_1 \times B'_1})| \geq \frac{c_1}{4}|A'_1||B'_1|$.

To obtain M_{v_2} we apply Lemma 3.1 again, now for $G_b|_{A'_1 \times B'_1}$. Here we can choose

$$c_1 = c_1^2 = \frac{c_1^1}{16} \quad \text{and} \quad c_2 = c_2^2 = \frac{c_2^1}{2\eta_1}.$$

M_{v_2} is the M_i guaranteed in the lemma. Note that technically this M_{v_2} is not the whole M_{v_2} in G_b, but it is restricted to $G_b|_{A'_1 \times B'_1}$. Denote $M_{v_2} = (A_2, B_2)$ where $A_2 \subset X_3, B_2 \subset X_4$.

We continue in this fashion. Assume that $M_{v_j} = (A_j, B_j)$ is already defined where $A_j \subset X_3, B_j \subset X_4$. Futhermore, we have $A'_j, B'_j \subset V(M_{v_j})$ such that

- $(A'_j \times B'_j) \cap M_{v_j} = \emptyset$,
- $|A'_j| = |B'_j| \geq \eta_j(|A'_{j-1}| + |B'_{j-1}|)$,
- $|E(G_b|_{A'_j \times B'_j})| \geq \frac{c_1^j}{4}|A'_j||B'_j|$.

To obtain $M_{v_{j+1}}$ we apply Lemma 3.1 for $G_b|_{A'_j \times B'_j}$. We can choose

$$c_1 = c_1^{j+1} = \frac{c_1^j}{16} \quad \text{and} \quad c_2 = c_2^{j+1} = \frac{c_2^j}{2\eta_j}.$$

$M_{v_{j+1}}$ is the M_i guaranteed in the lemma. Denote $M_{v_{j+1}} = (A_{j+1}, B_{j+1})$. We continue until $M_{v_1}, \dots, M_{v_{l_p}}$ are selected.

Next using these matchings M_{v_j} we choose a K_p such that it contains at most $q-1$ colors, a contradiction. For this purpose first we will find another sequence of matchings M'_{v_j} such that $M'_{v_j} \subset M_{v_j}$, $|M'_{v_j}| = a_j$ for $1 \leq j \leq l_p$.

$M'_{v_{l_p}}$ is just an arbitrary edge from $M_{v_{l_p}}$. Assume that $M'_{v_{l_p}}, \dots, M'_{v_{j+1}}$ are already defined and now we define M'_{v_j}. We consider the $2a_{j+1}$ vertices in $V(M'_{v_{j+1}})$ and the edges of M_{v_j} incident to these vertices. We have four cases.

Case 1. If $2a_{j+1} = a_j$, then this is M'_{v_j}.

Case 2. If $2a_{j+1} = a_j + 1$, so $a_j = b_k$ for an even k, then we remove one of the edges from this set incident to a vertex in $V(M'_{v_{j+1}}) \cap X_3$ to get M'_{v_j}. Furthermore, we mark this vertex in $V(M'_{v_{j+1}}) \cap X_3$ which is not covered by M'_{v_j}. This marked vertex is going to be covered only by $M'_{v_{j'}}$ if $a_{j'} = b_{k-1}$.

Case 3. If $2a_{j+1} = a_j - 1$ and there is no marked vertex at the moment, then to get M'_{v_j} we add one arbitrary edge of M_{v_j} to these $2a_{j+1}$ edges.

Case 4. Finally, if $2a_{j+1} = a_j - 1$ and there is a marked vertex then to get M'_{v_j} we add to these $2a_{j+1}$ edges the edge of M_{v_j} incident to the marked vertex and we "unmark" this vertex.

We continue in this fashion until $M'_{v_{l_p}}, \ldots, M'_{v_1}$ are defined. For a matching M'_{v_j} let us denote by $X_1(M'_{v_j})$ the set of those vertices in X_1 that form 4-edges in the 4-partite hypergraph $\mathcal{H}$ with the edges of M'_{v_j} and v_j. Observe that

$$\sum_{i=1}^{l_p} |X_1(M'_{v_i})| \leq \sum_{i=1}^{l_p} a_i, \tag{7}$$

since each M'_{v_i} could give rise to as many as a_i vertices in X_1.

Consider the following set S of vertices.

$$S = \bigcup_{j=1}^{l_p} (V(M'_{v_j}) \cup \{v_j\} \cup X_1(M'_{v_j})).$$

Here from the construction we have

$$\left| \cup_{j=1}^{l_p} V(M'_{v_j}) \right| \leq a_0. \tag{8}$$

Then using Lemma 2.4, Equations (4), (7), and (8) we get the following.

$$\begin{aligned} |S| &\leq a_0 + l_p + \sum_{j=1}^{l_p} a_j \leq 2a_0 - 2 + f_p(l_p) + l_p \leq 2a_0 - 1 + l_p \\ &\leq 2\left(\left\lfloor \frac{p}{2} \right\rfloor - \left\lceil \frac{\log_2 p}{2} \right\rceil \right) + \lceil \log_2 p \rceil - 2 < p. \end{aligned}$$

To get the desired K_p we add arbitrary vertices to S to have exactly p vertices. Then in this K_p by Lemma 2.4 the number of colors is at most

$$\begin{aligned} \binom{p}{2} - \sum_{j=1}^{l_p} a_j &= \binom{p}{2} - a_0 + 2 - f_p(l_p) \\ &\leq \binom{p}{2} - \left\lfloor \frac{p}{2} \right\rfloor + 2 + \left\lceil \frac{\log_2 p}{2} \right\rceil = q_{quad} + \left\lceil \frac{\log_2 p}{2} \right\rceil < q, \end{aligned}$$

a contradiction. This completes the proof of Theorem 1. ∎

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