Adding Algebraic Rewriting to the Untyped Lambda Calculus

Daniel J. Dougherty
Department of Mathematics
Wesleyan University
Middletown, CT 06457 USA
ddougherty@eagle.wesleyan.edu

Abstract

We investigate the system obtained by adding an algebraic rewriting system R to an untyped lambda calculus in which terms are formed using the function symbols from R as constants. On certain classes of terms, called here "stable", we prove that the resulting calculus is confluent if R is confluent, and terminating if R is terminating. The termination result has the corresponding theorems for several typed calculi as corollaries. The proof of the confluence result suggests a general method for proving confluence of typed β reduction plus rewriting; we sketch the application to the polymorphic lambda calculus.

1 Introduction

Term rewriting systems and the untyped lambda calculus are universal models of computation. Algebraic reduction is a natural technique for computing with standard functions such as successor and addition and with operations defined by equations over an abstract data type, while the lambda calculus has proven to be a powerful model of several aspects of modern programming languages (e.g., programmer-defined functions and their parameter passing mechanisms). It would seem profitable to combine the two modes, allowing each to do what it does best. For instance, as pointed out in [Bre88], algebraic rules such as rewriting $x - x$ to 0 could be treated as code optimizations in a functional language. From the point of view of the logic of programming, the equations from which rewriting rules are defined should allow the use of first order properties of the data to be involved in the higher order reasoning about programs.

The following example (from [BM88]) shows that the combination of algebra and untyped lambda calculus is problematic. Suppose we have a system which allows any term $x - x$ to be rewritten to 0, and a term $succ(x) - x$ to be rewritten to 1, and further suppose that terms have fixed points, so that there is a term X with X evaluating to $succ(X)$. Then $X - X$ evaluates to 0 and to 1.

The insight in Val Breazu-Tannen's [Bre88] is that restriction to various type disciplines should allow lambda terms to inherit nice properties from the algebraic system. (See also [BM88], [Bre87]). In [Bre88], it was shown that if a confluent algebraic system is added to the simply typed lambda calculus, the resulting system combining β and algebraic reductions is confluent. The question of preservation of termination was left open. Jean Gallier and Breazu-Tannen (independently of the present work) have shown that the polymorphic lambda calculus remains confluent when enriched by a confluent algebraic system ([BG89]). The same methods are used there to show that termination is inherited.

In this paper, we present some general results about the interaction between β-reduction and term rewriting. The proofs are purely syntactic, and they do not rely on any specific typing dicipline.

Let Σ be a set of function symbols with specified arities, fix a set $Vars = \{v_i | i \in \omega\}$ of variables, and let $\Lambda(\Sigma)$ be the set of lambda terms over $Vars \cup \Sigma$. Each algebraic term $f(A_1, \cdots, A_n)$ over Σ corresponds, via currying, to a $\Lambda(\Sigma)$-term $(fA_1 \cdots A_n)$, and so a system R of rewriting rules over Σ induces a rewriting relation $\xrightarrow{R}$ over $\Lambda(\Sigma)$. Write $\xrightarrow{\beta R}$ for $\xrightarrow{\beta} \cup \xrightarrow{R}$.

We will restrict attention to β-strongly normalizing terms in an attempt to avoid the difficulties arising from the existence of fixed points. But it is easy to see that preservation of termination in a combined system requires some other restriction on the lambda terms considered. For example, if the only algebraic rule is "$(fxx) \to x$", which is clearly terminating, then the β normal form $f(\lambda x.xx)(\lambda x.xx)(\lambda x.xx)$ is βR-infinite. A similar observation applies to confluence preservation. Furthermore, if the signature Σ does not specify arities for the function symbols, then even in the absence of β-reduction anomolies can occur when algebraic terms are curried. For example, the system R with rules $f(x) \to g(x,x)$ and $g(x) \to f(x,x)$ is trivially terminating in the algebraic setting, while $(gx) \to (fxx) \to (gxxx) \to \cdots$ is an infinite R-reduction on $\Lambda(\Sigma)$-terms. (I am indebted to the referee for the latter observation and example.)

We will not want to insist that function-arity is respected in the strictest sense, since we certainly wish to allow function symbols to occur (say, as arguments to higher-order procedures) without being instantiated by their arguments. But when the rewrite systen R is thought of as rewriting terms of base type, no function symbol should be presented with *more* arguments than its arity prescribes. Somewhat surprisingly, in the presence of β-strong normalization this very elementary form of type-checking, which we may call "arity-checking", will suffice to ensure inheritance of confluence or termination, without a commitment to a specific type discipline.

Definition 1.1 A set $\mathcal{S}$ of terms is *R-stable* if

1. $\mathcal{S}$ is closed under taking subterms and under $\xrightarrow{\beta R}$,
2. each term in $\mathcal{S}$ is β strongly normalizing, and contains no subterm of the form $(fA_1 \cdots A_n)$ where n is greater than the arity of f.

Examples of R-stable sets include the sets of $\Lambda(\Sigma)$ terms which receive a type in the simply typed lambda calculus, polymorphic lambda calculi ([Gir71], [Gir72], [Rey74]), certain systems of dependent types ([Mac86]), and the Calculus of Constructions ([CH88]). When an R and an R-stable set are available from the context we may abuse notation and speak of "stable terms".

Our main results are:

- If R is confluent then $\xrightarrow{\beta R}$ is confluent on R-stable terms.

- If R is terminating then $\xrightarrow{\beta R}$ is terminating on R-stable terms.

The constructions can be briefly described as follows. To show preservation of confluence, we follow Breazu-Tannen [Bre88] in projecting βR-reduction to R-reduction on β-normal forms, but simplify and generalize his technique by passing to a bottom-up/parallel version of R-reduction which almost commutes, in a technical sense, with β-reduction. This relation is similar to the relation used by Tait and Martin-Löf in their proof of β-confluence for untyped lambda calculus. To analyze termination, we show that whenever M is not a β-normal form then any βR-reduction out of M can be *projected* along a properly chosen β-reduction in such a way that if the first reduction is infinite then so is the projection. We will be able to conclude that if M allows an infinite βR-reduction then the β-normal form of M will allow an infinite R-reduction – a contradiction if R is assumed to be a terminating rewrite system.

A consequence of the approach we have adopted is that in order to derive the corresponding theorems about typed systems, the statements of the theorems above are not sufficient (this is a familiar phenomenon; consider the confluence of the simply typed lambda calculus, which will not follow from the fact that the untyped lambda calculus is confluent, but which submits to exactly the same proof). By making essentially trivial modifications to the *proofs* of the theorems given here, the reader may derive termination- and confluence-preservation results for simply typed and polymorphically typed lambda calculi. We omit a detailed treatment, but we do outline the the modifications needed to address the Girard-Reynolds system $\lambda^{\forall}$.

In [Klo80], Klop considers the addition of new rewriting rules to untyped lambda calculus, with restrictions on the form of the new rules (for example, that variables may not occur twice on the left side of a rule). We treat arbitrary algebraic rules. Toyama, in [Toy87], shows that the direct sum of confluent term rewriting systems is confluent, but the purely algebraic setting is very different from the present framework.

The termination of a combination of terminating algebraic rewrite systems is a very delicate issue – Toyama presents several counterexamples in [Toy87]. Termination is known to persist in a combined system under various, somewhat restrictive, hypotheses ([Mid89], [Rus87], [TKB89]).

We note in passing that the addition of constants and rules expressing surjective pairing on untyped terms can be cast as rewrite rules as above, but these are explicitly higher-order constructs and do not conform to our notion of "algebraic". It is well-known ([Klo80]) that the addition of these rules to the untyped calculus disturbs confluence, and that for simply typeable terms, confluence and termination are each preserved ([Pot81]). However, the techniques of the present paper do not shed any light on this situation, since arity-checking fails for simply typed terms with pairing. It may be that there is a generalization of the notion of stability which allows the present techniques to be adapted to such non-algebraic reduction.

For basic definitions and notation not explained below, see [Bar84] for the lambda calculus, and [HO80] for term rewriting.

Notation

Following a convention of algebraic rewriting, we identify each term with a partial function on sequences (here, $\{0,1\}$-sequences). Thus, a variable or constant is identified with the function mapping the empty sequence to that variable or constant; (A_0A_1) maps the empty sequence to a symbol for application, and maps the sequence iu to $A_i(u), i \in \{0,1\}$; and (λxA) maps the empty sequence to a symbol for "λx" and maps $0u$ to $A(u)$. Sequences in the domain of a term A are called *occurrences* in A. If a and b are sequences with a an initial segment of b, we say that a *precedes* b and that b *extends* a; a and b are *incomparable* if neither extends the other.

Suppose u is an occurrence in A. Then A/u is the term defined by $A/u(v) = A(uv)$; if B is of the form A/u, then B is a *subterm* of A (written $B \sqsubseteq A$). The term $A[u \leftarrow B]$ is defined by: $A[u \leftarrow B](w) = A(w)$ for occurrences w of A not extending u and $A[u \leftarrow B](uv) = B(v)$ for occurrences v of B.

Any term is in one of the forms:

- $\lambda x_1 \cdots x_n.hM_1 \cdots M_m$, with $n, m \geq 0$ and $h \in Vars \cup \Sigma$, or
- $\lambda x_1 \cdots x_n.(\lambda x M_0)M_1 \cdots M_m$, with $n \geq 0$ and $m \geq 1$.

A Σ term is either a variable or a term $(fA_1 \cdots A_m)$, $f \in \Sigma, A_i \in \Lambda(\Sigma), (0 \leq i \leq m)$. An *algebraic* term is either a variable or a term $(fT_1 \cdots T_m)$, $f \in \Sigma$, $m =$ the arity of f , T_i algebraic for $(0 \leq i \leq m)$. We will reserve S and T to stand for algebraic terms.

Substitution into an algebraic term T is particularly simple since there is no variable binding in T. Since we will often have occassion to manipulate substitution instances of algebraic terms T, we adopt the following notational convention:

> *If T is an algebraic term whose free variables are among the set $\{v_1, \ldots, v_k\}$, and $\vec{Q} \equiv \{Q_1, \ldots, Q_k\}$ is any multiset of $\Lambda(\Sigma)$ terms, then the result of simultaneously substituting each Q_i for v_i in T is denoted $T^{\vec{Q}}$.*

An *algebraic rewrite system* R is a set of pairs $\langle S, T\rangle$ of algebraic terms in which S is not a variable and $Vars(T) \subseteq Vars(S)$.

Let P be any binary relation on terms. A *P-reduction step* ρ from M to N is given by a pair $\langle A, B\rangle \in P$, an occurrence u of M, and a substitution σ such that $M/u \equiv \sigma(A)$ and $N \equiv M[u \leftarrow \sigma(B)]$. We say that ρ has *source term* A, and *redex* u. Any term $\sigma(A)$ as above is called a *P-redex term.* The relation $\xrightarrow{P}$ holds from M to N if there is a P-reduction ρ from M to N (we write $\rho : M \xrightarrow{P} N$).

Given $\xrightarrow{P}$, we denote its reflexive closure by $\xrightarrow{P}_{\equiv}$, its reflexive transitive closure by $\xrightarrow{P}\!\!\!\!\rightarrow$, and its associated equivalence relation by $\twoheadleftarrow\!\!\!\xrightarrow{P}\!\!\!\twoheadrightarrow$. If we write $\rho : M_0 \xrightarrow{P} M_1 \xrightarrow{P} \cdots$ then ρ is a sequence of reduction steps $\langle \rho_i | 0 \leq i < n\rangle$ for some $n \leq \omega$.

A relation P is *confluent*, or *Church-Rosser*, if whenever $X \overset{P}{\twoheadleftarrow} M \overset{P}{\twoheadrightarrow} Y$ there is an N such that $X \overset{P}{\twoheadrightarrow} N \overset{P}{\twoheadleftarrow} Y$. P is *terminating*, or *strongly normalizing*, if there are no infinite $\xrightarrow{P}$ reductions.

If P and Q are relations, we often write PQ for $P \cup Q$.

2 Descendants and Projections

The proofs of our preservation theorems proceed by isolating certain subterms of a term and analyzing reductions into steps which take place inside the given terms and other steps which are blind to the internal structure of those terms. In this section we develop some machinery enabling us to track the progress of subterms during a reduction. We use the notions of *descendant* of an occurrence with respect to an algebraic reduction (essentially as in [HL79]).

Definition 2.1 Let $\rho : M \xrightarrow{R} N$ have redex u and source term S.

For an occurrence d of M, the set d/ρ of *descendants of* d *with respect to* ρ is the set of occurrences in N defined as follows. If d does not extend u then $d/\rho = \{d\}$. If d is uw, w a non-variable address of S, then $d/\rho = \emptyset$. Otherwise, writing d as uac, where $S/a \equiv v_i$, d/ρ is $\{ua'c | T/a' = v_i\}$.

If $\mathcal{D}$ is a set of occurrences in M then $\mathcal{D}/\rho$ is $\bigcup\{d/\rho | d \in \mathcal{D}\}$.

If $\rho = \rho_n \circ \cdots \circ \rho_0$ is a several step reduction then $\mathcal{D}/\rho$ is $(\cdots(\mathcal{D}/\rho_0)/\cdots/\rho_n)$.

When the descendant of a certain occurrence of subterm X is under consideration, we will often simply say "descendant of X". For example, suppose $\langle fx, gxx\rangle$ is a rule, and consider the reduction $M \equiv h(f(ky)) \xrightarrow{R} h(g(ky)(ky)) \equiv N$. Then ky has two descendants, the two occurrences of ky in N, and $f(ky)$ has one descendant, *viz.*, $g((ky)(ky))$ in N.

We will pay particular attention to maximal non-Σ subterms, the occurrences of which form the Σ-*boundary* of a term.

Definition 2.2 The Σ-*boundary* of M, $O_\Sigma(M)$, is the set of occurrences defined as follows (by induction on terms).

- If M is a variable then $O_\Sigma(M)$ is empty.
- If M is of the form $fM_1 \cdots M_m$ with $f \in \Sigma$, then d is in $O_\Sigma(M)$ iff $\exists i\ \exists d_1\ \exists d_2,\ d = d_1 d_2,\ M/d_1 \equiv M_i,\ d_2 \in O_\Sigma(M_i)$.
- Otherwise $O_\Sigma(M)$ contains the empty occurrence only (corresponding to the term M itself).

Lemma 2.3 *If T is algebraic then d is in $O_\Sigma(T^{\vec{Q}})$ iff $\exists i\ \exists d_1\ \exists d_2,\ d = d_1 d_2,\ T^{\vec{Q}}/d_1 \equiv Q_i,\ d_2 \in O_\Sigma(Q_i)$.*

Proof. An easy induction on T. □

Lemma 2.4 *Let $\rho : M \xrightarrow{R} N$, and let d be an occurrence in M.*

1. *For each $e \in d/\rho$, $M/d \xrightarrow{R}_{\equiv} N/e$.*
2. *If d precedes the redex of ρ then $M/d \xrightarrow{R} N/d$.*
3. *If M is stable and $\mathcal{D}$ is $O_\Sigma(M)$ then $\mathcal{D}/\rho$ is $O_\Sigma(N)$.*

Proof. The first two assertions follow easily from the definition of descendant.

For part 3, use induction on M. If M is of the form $\lambda x_1 \cdots x_n.(\lambda x M_0) M_1 \cdots M_m$, $(n \geq 0, m \geq 1)$, then N is obtained by a rewrite inside one of the M_i, and the induction hypothesis applies (since the same clause in the definition of Σ-boundary applies for N). The same argument applies if M is $\lambda x_1 \cdots x_n.hM_1 \cdots M_m$, unless $n = 0$ and the redex of ρ is not contained in some M_i. But in this case stability (arity-checking in particular) implies that $M \equiv hM_1 \cdots M_m$ is in fact $S^{\vec{Q}}$ and N is $T^{\vec{Q}}$, for some rule $\langle S, T\rangle$ and some $\vec{Q}$. The result then follows from Lemma 2.3 and the fact that no variables occur in T which are not in S. □

To isolate the steps of a reduction which are independent of some particular subterms, we consider the term obtained by replacing those subterms by variables. We must do this with some care in order to preserve the rewriting relation.

Definition 2.5 An *R-projection* (or just *projection*, if R is available from the context) is any function π from terms to variables such that if M and N have a common R-reduct then they are assigned the same variable.

Given a set $\mathcal{D}$ of pairwise incomparable occurrences in a term M, and a projection π whose range is disjoint from the variables of M, write M^π for the term obtained from M by replacing M/d by $\pi(M/d)$ for each $d \in \mathcal{D}$, and say that M^π is a *projection of M at $\mathcal{D}$*.

If $\rho : M \xrightarrow{R} N$, π is a projection at $\mathcal{D}$, and $\mathcal{D}/\rho$ is pairwise incomparable, then $\rho\pi$ is the projection at $\mathcal{D}/\rho$ given by $\rho\pi(N/e) \equiv \pi(M/d)$, for each $d \in \mathcal{D}$ and $e \in d/\rho$.

We need Lemma 2.4.1 in order to justify the definition of $\rho\pi$ above. In order to ensure that $\mathcal{D}/\rho$ is pairwise incomparable, it will suffice (by Lemma 2.4.3) to choose $\mathcal{D}$ to be the Σ-boundary of M (it is clear that a Σ-boundary is a pairwise incomparable set of occurrences). To go further and have the projection of a reduction induce a reduction on the projections, we must be careful to project on a sufficiently full set of occurrences, in the following sense:

Definition 2.6 If M/u is of the form $T^{\vec{Q}}$ and $\mathcal{D}$ is a set of occurrences, then $\mathcal{D}$ is (T, u) *full* if no $d \in \mathcal{D}$ is uw with w a non-variable occurrence in T, and for every $d \in \mathcal{D}$ which is uac with T/a a variable, $\mathcal{D}$ contains each $ua'c$ for which T/a' is the same variable.

Lemma 2.7 *Let $\rho : M \xrightarrow{R} N$, have redex u and source term S, let $\mathcal{D}$ be pairwise incomparable and (S, u) full, and suppose that M^π is a projection of M at $\mathcal{D}$. Then*

1. $M^\pi \xrightarrow{R}_{\equiv} N^{\rho\pi}$.
2. *If no $d \in \mathcal{D}$ precedes u, then $M^\pi \xrightarrow{R} N^{\rho\pi}$.*

Proof. Let M/u be $S^{\vec{Q}}$ for some $\vec{Q}$. If no $d \in \mathcal{D}$ precedes u then u is an occurrence in M^π. But since $\mathcal{D}$ is (S, u) full, M^π/u is of the form $S^{\vec{Q}'}$ and $N^{\rho\pi}/u$ is of the form $T^{\vec{Q}'}$, proving (2).

On the other hand, if there is a $d_0 \in \mathcal{D}$ preceding u, then $\mathcal{D}/\rho = \mathcal{D}$ and $M^\pi \equiv N^{\rho\pi}$. So $M^\pi \xrightarrow{R}_{\equiv} N^{\rho\pi}$ holds in any case. □

Finally, in order to project an R reduction of several steps on $\mathcal{D}$ we must guarantee that its descendants will be full for the next step. This motivates the next result.

Lemma 2.8 *If $\mathcal{D}$ is $O_\Sigma(M)$ then for any u such that M/u is of the form $T^{\vec{Q}}$, $\mathcal{D}$ is (T,u)-full.*

Proof. Let $d \in \mathcal{D}$ be uac and let $d' = ua'c$, with T/a and T/a' the same variable; it suffices to check that there can be no non-Σ occurrence above d'. Indeed, if v' were one, then we could find one above d as follows. Either v' would be above u, hence itself above d, or below a', in which case there would be a v below a and above d with $M/v' \equiv M/v$. This contradicts $d \in \mathcal{D}$. □

Thus Σ-boundaries are always sets of non-Σ occurrences which are sufficiently full, and in the stable case their descendants inherit this property. These facts will enable us to iterate applications of Lemma 2.7 when we start with a projection of a Σ-boundary.

It will be important to isolate β-redex subterms of a term which are contained in no other β-redexes, and whose descendants are similarly maximal. Leftmost redexes have these properties under β-reduction alone, but algebraic reduction can spoil leftmost-ness. So we need a generalization:

Definition 2.9 An occurrence d is an *outermost* β-redex occurrence of M if either

- $M \equiv \lambda x_1 \cdots x_n.(\lambda x M_0)M_1 \cdots M_m$, and d is the indicated occurrence of $(\lambda x M_0)M_1$, or
- $M \equiv \lambda x_1 \cdots x_n.hM_1 \cdots M_m$ with $h \in \mathit{Vars} \cup \Sigma$, and $\exists i\ \exists d_1\ \exists d_2,\ d = d_1 d_2,\ M/d_1 \equiv M_i,\ d_2$ outermost in M_i.

Of course, outermost β redexes need not be in the Σ-boundary of M.

Lemma 2.10 1. *If T is algebraic then d is an outermost β-redex in $T^{\vec{Q}}$ iff $\exists i\ \exists d_1\ \exists d_2,\ d = d_1 d_2,\ T^{\vec{Q}}/d_1 \equiv Q_i,\ d_2$ outermost in $O_\Sigma(Q_i)$.*

2. *When M is stable, each R-descendant of an outermost β-redex is an outermost β-redex.*

3. *When M is a stable β-normal form each R-reduct of M is a β-normal form.*

Proof.

1. An easy induction on T.

2. By induction on M; suppose $M \xrightarrow{R} N$. If $M \equiv \lambda x_1 \cdots x_n.(\lambda x M_0)M_1 \cdots M_m$ then N has the same shape and the result is clear. When $M \equiv \lambda x_1 \cdots x_n.hM_1 \cdots M_m$, proceed using induction and part 1, (in the same manner as in the proof of Lemma 2.4.3).

3. By induction on M. We may write $M \equiv \lambda x_1 \cdots x_n.hM_1 \cdots M_m \xrightarrow{R} N$. The only case which does not submit immediately to the induction hypothesis is the one in which $M \equiv hM_1 \cdots M_m$ is $S^{\vec{Q}}$ and $N \equiv T^{\vec{Q}}$. But part 1 and the fact that there are no variables in T which are not in S imply that there can be no (outermost) β-redexes in N.

□

3 Termination

In this section it will be shown that if a terminating algebraic rewriting system is added to the $\Lambda(\Sigma)$ calculus of β-reduction, the resulting system is terminating on stable terms. We assume that all terms under consideration are stable.

The first step is to record some well-known results on β-reduction which parallel some of the results of the previous section. The notions of *residual* of a β-redex and of a *development* of a specified set of redexes are standard, and it turns out that we can confine our attention to developing sets of incomparable β-redexes. In the interest of maintaining a uniform terminology we will use "descendant" to refer to the image of an occurrence under either type of reduction. Hence:

Notation 3.1

- If $\rho : M \xrightarrow{\beta}\!\!\!\!\rightarrow \cdots$, and $\mathcal{D}$ is a set of β-redex occurrences in M, then the set $\mathcal{D}/\rho$, of descendants of $\mathcal{D}$ with respect to ρ is the set of occurrences of residuals of the terms at $\mathcal{D}$.
- Let $\mathcal{D}$ be a set of pairwise incomparable β-redexes in M. Then $\varphi(\mathcal{D}\ , M)$ is the term obtained from M by contracting those redexes. We say that $\varphi(\mathcal{D}\ , M)$ is a *development* of M.

Lemma 3.2 *Let $\rho : M \xrightarrow{\beta} N$, and let d be an outermost β-redex occurrence in M.*

1. *If d is the redex of ρ then $d/\rho = \emptyset$, otherwise $d/\rho = \{d\}$.*
2. *$M/d \xrightarrow{\beta}_{\equiv} N/d$.*
3. *If d strictly precedes the redex of ρ, then $M/d \xrightarrow{\beta} N/d$.*
4. *Each β descendant of an outermost β-redex is an outermost β-redex.*

Proof. The first three parts are clear. The proof of part 4 is an induction on terms. Suppose M is $\lambda x_1 \cdots x_n.(\lambda x M_0)M_1 \cdots M_m$. If the redex of ρ is $(\lambda x M_0)M_1$ itself there are no descendants; otherwise the descendant of $(\lambda x M_0)M_1$ is in the same position in N and hence is outermost. When M is $\lambda x_1 \cdots x_n.hM_1 \cdots M_m$, apply induction. $\square$

The construction in the proof of termination for βR involves choosing an outermost β-redex from the initial term of a reduction and developing it and all of its descendants. The next two results show that under the right conditions, such a development preserves $\xrightarrow{\beta}\!\!\!\!\rightarrow$ and $\xrightarrow{R}\!\!\!\!\rightarrow$. The first is a special case of the strong theorem (FD!) on finite developments.

In a β redex term $(\lambda x P)Q$, call Q the *argument term.* Note that if D is a β redex subterm of Q in $(\lambda x P)Q$, and if furthermore $x \notin FV(P)$, then a contraction of D is rendered moot by a subsequent contraction of $(\lambda x P)Q$. This possibility plays a role in the next two lemmas.

Lemma 3.3 *Let $\rho : M \xrightarrow{\beta} N$. If $\mathcal{D}$ is a set of outermost β redexes in M, and $\mathcal{E}$ is $\mathcal{D}/\rho$, then*

1. *$\varphi(\mathcal{D}, M) \xrightarrow{\beta}\!\!\!\!\rightarrow \varphi(\mathcal{E}, N)$, and*

2. *if the redex of ρ is neither an element of $\mathcal{D}$ nor an occurrence in an argument term of a redex from $\mathcal{D}$ then at least one reduction is done in $\varphi(\mathcal{D}, M) \xrightarrow{\beta}\!\!\!\!\rightarrow \varphi(\mathcal{E}, N)$.*

Proof.

1. This is Lemma 11.1.7.(ii) of [Bar84].

2. Assuming the redex u of ρ is as described, u has at least one descendant relative to any reduction from M to $\varphi(\mathcal{D}, M)$. With this observation, the proof is, verbatim, the proof of Lemma 11.3.3 of [Bar84].

□

Now, if $\mathcal{D}$ is a set of outermost β-redexes and ρ is an algebraic reduction, then we know that the descendants are also outermost (hence incomparable) β redexes, so it makes sense to develop them. This leads to an algebraic companion to Lemma 3.3, describing the interaction between algebraic reductions and β developments.

Lemma 3.4 *Let M be stable and $\rho : M \xrightarrow{R} N$. If $\mathcal{D}$ is a set of outermost β redexes in M and $\mathcal{E}$ is $\mathcal{D}/\rho$, then*

1. *$\varphi(\mathcal{D}, M) \xrightarrow{\beta R}\!\!\!\!\rightarrow \varphi(\mathcal{E}, N)$, and*

2. *if the redex of ρ is not an occurrence in an argument term of a redex from $\mathcal{D}$ then at least one reduction is done in $\varphi(\mathcal{D}, M) \xrightarrow{\beta R}\!\!\!\!\rightarrow \varphi(\mathcal{E}, N)$.*

Proof. Let ρ have redex u and source term S.

1. We have two cases, defined by the position of u with respect to $\mathcal{D}$ (of course $u \notin \mathcal{D}$). If no d in $\mathcal{D}$ precedes u, expand $\mathcal{D}$ to the smallest (S, u) full set $\mathcal{D}^+$ containing $\mathcal{D}$. Then $\mathcal{D}^+$ is still a set of outermost β redexes (cf. Lemma 2.10.1) and $\mathcal{D}^+/\rho$ is $\mathcal{E}$. Now if π is any projection on $\mathcal{D}^+$, Lemma 2.7 implies that $M^\pi \xrightarrow{R} N^{\rho\pi}$. Therefore by performing β reductions in M before the R reduction we obtain $\varphi(\mathcal{D}, M) \xrightarrow{\beta}\!\!\!\!\rightarrow \varphi(\mathcal{D}^+, M) \xrightarrow{R} \varphi(\mathcal{E}, N)$.

   If the redex u extends some $d_0 \in \mathcal{D}$, then no element of $\mathcal{D}$ extends u, since they are all incomparable with d_0. Therefore $\mathcal{D}$ is trivially (S, u) full, each element of $\mathcal{D}$ is its own descendant, and we have two subcases as follows. Write $M/d_0 \equiv (\lambda x A)B$, and $M/u \equiv S^{\vec{Q}}$.

   - when $S^{\vec{Q}} \sqsubseteq A$,

   $$M \equiv M[d_0 \leftarrow (\lambda x A[u \leftarrow S^{\vec{Q}}])B],$$

   $$N \equiv M[d_0 \leftarrow (\lambda x A[u \leftarrow T^{\vec{Q}}])B],$$

   $$\varphi(\mathcal{D}, M) \equiv M'[d_0 \leftarrow (A[u \leftarrow S^{\vec{Q}}])[x := B]],$$

   $$\varphi(\mathcal{E}, N) \equiv M'[d_0 \leftarrow (A[u \leftarrow T^{\vec{Q}}])[x := B]],$$

   and $\varphi(\mathcal{D}, M) \xrightarrow{R} \varphi(\mathcal{E}, N)$ by substitutivity of R.

- When $S^{\vec{Q}} \sqsubseteq B$,

$$M \equiv M[d_0 \leftarrow (\lambda x A)(B[u \leftarrow S^{\vec{Q}}])],$$
$$N \equiv M[d_0 \leftarrow (\lambda x A)(B[u \leftarrow T^{\vec{Q}}])],$$

$$\varphi(\mathcal{D}, M) \equiv M'[d_0 \leftarrow A[x := B[u \leftarrow S^{\vec{Q}}]]],$$

$$\varphi(\mathcal{E}, N) \equiv M'[d_0 \leftarrow A[x := B[u \leftarrow T^{\vec{Q}}]]],$$

  and $\varphi(\mathcal{D}, M) \xrightarrow{R}\!\!\!\!\!\rightarrow \varphi(\mathcal{E}, N)$ by repeating the R-reduction for every free occurrence of x in A.

2. The second assertion can be seen by examining the cases in part 1 – the only case where collapsing might occur is in the last case, when x is not free in A.

□

We are now in a position to see that βR-reduction is terminating on stable terms. It is convenient to treat pure R-reduction first.

Theorem 3.5 *If R is terminating on algebraic terms, then R is terminating on R-stable $\Lambda(\Sigma)$ terms.*

Proof. For the sake of contradiction, let M be a stable term of minimal size among those which are R-infinite.

By hypothesis, M cannot be algebraic. Suppose M were not a Σ term. Then M would be one of $xP_1 \cdots P_n$, or $(\lambda x.P_1)P_2 \cdots P_n$, $(n > 0)$, each R-reduct would be of the same shape, so that some P_i would be R-infinite, contradicting the minimality of M.

So let $\mathcal{D}$ be $O_\Sigma(M)$ and let π be a projection on $\mathcal{D}$ which replaces all subterms by the same variable. Since M is a Σ term, each subterm represented in $\mathcal{D}$ is smaller than M, and since M is not algebraic, M^π is smaller than M.

Now let $\rho : M_0 \xrightarrow{R} M_1 \xrightarrow{R} \cdots$ be an infinite R reduction, set $\mathcal{D}_0 \equiv \mathcal{D}$, $\mathcal{D}_{n+1} \equiv \mathcal{D}_n/\rho_n$, set $\pi_0 \equiv \pi$, $\pi_{n+1} \equiv \rho_n\pi_n$ and construct the sequence of terms $M_n^{\pi_n}$. By Lemma 2.7.1, $M_n^{\pi_n} \xrightarrow{R}_{\equiv} M_{n+1}^{\pi_{n+1}}$ for each n.

Since M^π is smaller than M, the sequence above is finite as a reduction sequence, so that for some k, $M_n^{\pi_n} \xrightarrow{R} M_{n+1}^{\pi_{n+1}}$ fails for all $n \geq k$.

For $n \geq k$, Lemma 2.7.2 applied to the reduction $\rho_n : M_n \xrightarrow{R} M_{n+1}$ yields a $d_n \in \mathcal{D}_n$ preceding the redex of ρ_n. It follows that for $n \geq k$, $\mathcal{D}_n/\rho_n = \mathcal{D}_k$. Furthermore, there must be a particular $d \in \mathcal{D}_k$ such that for infinitely many n, d precedes the redex of ρ_n. Thus M/d must be $\xrightarrow{R}$ infinite, contradicting the minimality of M. □

Theorem 3.6 *If R is terminating on algebraic terms, then βR is terminating on R-stable $\Lambda(\Sigma)$ terms.*

Proof. The proof is by induction on the maximum number of steps which can occur in a β-reduction of a stable term M. For the sake of contradiction, let $\rho : M \equiv M_0 \stackrel{\beta R}{\longrightarrow} M_1 \stackrel{\beta R}{\longrightarrow} \cdots$ be an infinite reduction.

When M is a β-normal form, Lemma 2.10.3 implies that each ρ_n is an R-reduction, so ρ is finite by Theorem 3.5.

So let d_0 be the leftmost β-redex in M_0, $M/d_0 \equiv (\lambda x P_0)Q_0$. This is certainly outermost. Since stability is inherited by subterms, the induction hypothesis applies to Q_0, so Q_0 is βR terminating.

Set $\mathcal{D}_0 = \{d_0\}, \mathcal{D}_{n+1} = \mathcal{D}_n/\rho_n$. Each $\mathcal{D}_n$ is a set of outermost β redexes by Lemmas 2.4.3 and 3.2.4, hence is pairwise incomparable. Lemmas 3.3.1 and 3.4.1 imply that $\varphi(\mathcal{D}_n, M_n) \stackrel{\beta R}{\twoheadrightarrow} \varphi(\mathcal{D}_{n+1}, M_{n+1})$ for each n, but by induction, $\varphi(\{d_0\}, M)$ is βR-terminating, so this is finite as a $\stackrel{\beta R}{\longrightarrow}$ reduction.

By Lemmas 3.3.2 and 3.4.2, from some point on each ρ_n-redex term is either equal to some β-redex term from $\mathcal{D}_n$, or is a subterm of the argument part of such a term. A reduction ρ_n of the first type results in $\mathcal{D}_{n+1}$ being smaller than $\mathcal{D}_n$, while one of the second type yields $\mathcal{D}_{n+1}$ the same size as $\mathcal{D}_n$, so eventually every reduction is of the second type. That is, there is a k such that for $n \geq k$ each ρ_n has its redex term inside the Q of some term $(\lambda x P)Q$ occurring in $\mathcal{D}_n$. Just as in the previous theorem, for $n \geq k$, $\mathcal{D}_n/\rho_n = \mathcal{D}_k$, and there is a particular $d \in \mathcal{D}_k$ such that for infinitely many n, d precedes the redex of ρ_n.

Now, M_k/d is of the form $(\lambda x P_k)Q_k$, $(\lambda x P_0)Q_0 \stackrel{\beta R}{\twoheadrightarrow} (\lambda x P_k)Q_k$, and no step in this reduction occurs at the root of a term, so in fact $Q_0 \stackrel{\beta R}{\twoheadrightarrow} Q_k$. The previous paragraph showed that Q_k is βR-infinite, so we have a contradiction of the fact that Q_0 is βR terminating. □

It follows that adding terminating algebraic rules to the simply or polymorphically typed lambda calculus results in a terminating system, since an infinite reduction in the calculus would induce an infinite reduction in the βR system defined by erasing the types from terms. A different proof of this fact, obtained independently, is found in [BG89] .

Indeed, any strongly normalizing typed lambda calculus which admits a notion of type erasure so that β reductions induce β reductions on the untyped erasures, and in which there can be no infinite βR reductions which are invisible to the erasures, (for example, within the types themselves), will remain strongly normalizing if a terminating set of rules is added.

4 Confluence

This section shows that when confluent algebraic rewriting is combined with β reduction, confluence is inherited by stable terms. We again restrict attention to stable terms.

As pointed out in [Bre88], we cannot expect confluence in the presence of η reduction: if $fx \stackrel{R}{\longrightarrow} a$, then $\lambda x.fx$ has the two ηR normal forms $\lambda x.a$ and f.

We first verify that a confluent algebraic system R remains confluent when extended to the expanded set of terms $\Lambda(\Sigma)$. The global strategy in the proof of Theorem 4.1 (projecting βR-reductions to R-reductions on β-normal forms) was used in [Bre88] in the simply typed setting; we avoid the use of types in the argument.

Theorem 4.1 *If R is confluent on algebraic terms, then R is confluent on R-stable $\Lambda(\Sigma)$ terms.*

Proof. We show by induction on stable terms M that if $X \stackrel{R}{\twoheadleftarrow} M \stackrel{R}{\twoheadrightarrow} Y$ then there exists N such that $X \stackrel{R}{\twoheadrightarrow} N \stackrel{R}{\twoheadleftarrow} Y$. If M is algebraic, confluence holds by hypothesis. If M is itself not a Σ term, then M can be written as one of $xP_1 \cdots P_n$ or $(\lambda x.P_1)P_2 \cdots P_n$, $(n > 0)$, and X and Y must have the same shape, so we can build N using the induction hypothesis on the P_i.

So suppose M is a non-algebraic Σ term, let $\mathcal{D}$ be $O_\Sigma(M)$, and let R^- be the relation R restricted to $\{A | \exists d \in \mathcal{D}, M/d \stackrel{R}{\twoheadrightarrow} A\}$. Since M is a Σ term, each M/d for $d \in \mathcal{D}$ is smaller than M. By the induction hypothesis, R confluence holds out of M/d when $d \in \mathcal{D}$, and it follows that R confluence holds out of every term in the domain of R^-. So R^- is a confluent relation.

Let π be defined over $\mathcal{D}$ so that terms M/d and M/e are replaced by the same variable if and only if $M/d \stackrel{R^-}{\twoheadleftarrow\!\!\twoheadrightarrow} M/e$. This is an R-projection. Since M is not algebraic, M^π is smaller than M.

By iterating Lemma 2.7 we can project the two reductions on $\mathcal{D}$ and its descendants, obtaining $M^\pi \stackrel{R}{\twoheadrightarrow} X'$ and $M^\pi \stackrel{R}{\twoheadrightarrow} Y'$. By the induction hypothesis applied to M^π there exists N' with $X' \stackrel{R}{\twoheadrightarrow} N' \stackrel{R}{\twoheadleftarrow} Y'$.

The terms N', X', and Y' are obtained from M, X, and Y respectively by replacing subterms by new variables. We build our desired N by finding appropriate terms to substitute for these variables in N'.

Consider one of the new variables z and let $A_1, \ldots A_l$ be the subterms of X replaced by z to give X', and $B_1, \ldots B_m$ the subterms of Y replaced by z to give Y'. Each A_i is a reduct of some M/d with $d \in \mathcal{D}$, and the same holds for each B_j. By the confluence of R^-, we can produce a term $C_{(z)}$ which is a common R-reduct of each A_i and B_j.

When this has been done for each z, take N to be N' with each z replaced by $C_{(z)}$.

Now we have $X \stackrel{R}{\twoheadrightarrow} X'[\vec{z} := \vec{C}_{(z)}]$ by rewriting the various A_i to $C_{(z)}$ for each z and A_i as above. Similarly, $Y \stackrel{R}{\twoheadrightarrow} Y'[\vec{z} := \vec{C}_{(z)}]$. Finally, $X'[\vec{z} := \vec{C}_{(z)}] \stackrel{R}{\twoheadrightarrow} N'[\vec{z} := \vec{C}_{(z)}]$ and $Y'[\vec{z} := \vec{C}_{(z)}] \stackrel{R}{\twoheadrightarrow} N'[\vec{z} := \vec{C}_{(z)}]$ by substitutivity of R. Thus $N'[\vec{z} := \vec{C}_{(z)}]$ is the desired N. $\Box$

To lift this result to full βR reduction, we attempt to project reductions to reductions on β normal forms (the latter reductions will be purely algebraic if the original term is stable). Now, R reductions will not commute directly with β reductions, but the relation $\stackrel{R}{\twoheadrightarrow}_1$ defined below almost commutes with $\stackrel{\beta}{\longrightarrow}$. The technique is inspired by a proof of the confluence of $\stackrel{\beta}{\longrightarrow}$ due to Tait and Martin-Löf.

Definition 4.2 The relation $\stackrel{R}{\twoheadrightarrow}_1$ is defined inductively as follows:

1. $M \stackrel{R}{\twoheadrightarrow}_1 M$.
2. If $M \stackrel{R}{\twoheadrightarrow}_1 M'$ and $N \stackrel{R}{\twoheadrightarrow}_1 N'$ then $MN \stackrel{R}{\twoheadrightarrow}_1 M'N'$.
3. If $M \stackrel{R}{\twoheadrightarrow}_1 M'$ then $\lambda x.M \stackrel{R}{\twoheadrightarrow}_1 \lambda x.M'$.

4. If $\langle S,T\rangle \in R$, and for $1 \le i \le n$, $P_i \xrightarrow{R}\!\!\!\!\rightarrow_1 Q_i$, then $S^{\vec{P}} \xrightarrow{R}\!\!\!\!\rightarrow_1 T^{\vec{Q}}$.

The relation $\xrightarrow{R}\!\!\!\!\rightarrow_1$ is sometimes known as the *walk* relation.

Lemma 4.3 $\xrightarrow{R} \subseteq \xrightarrow{R}\!\!\!\!\rightarrow_1 \subseteq \xrightarrow{R}\!\!\!\!\rightarrow$.

Proof. The first is clear from (1) and (4) of Definition 4.2, the second is an easy induction over $\xrightarrow{R}\!\!\!\!\rightarrow_1$.

Lemma 4.4 *$A \xrightarrow{R}\!\!\!\!\rightarrow_1 B$ implies $M[x := A] \xrightarrow{R}\!\!\!\!\rightarrow_1 M[x := B]$.*

Proof. An easy induction on M.

Lemma 4.5 *$A \xrightarrow{R}\!\!\!\!\rightarrow_1 B$ and $M \xrightarrow{R}\!\!\!\!\rightarrow_1 N$ imply $M[x := A] \xrightarrow{R}\!\!\!\!\rightarrow_1 N[x := B]$.*

Proof. By induction on the derivation of $M \xrightarrow{R}\!\!\!\!\rightarrow_1 N$. When $M \equiv N$, use the previous lemma. The cases when $M \xrightarrow{R}\!\!\!\!\rightarrow_1 N$ follows from parts 2 or 3 of Definition 4.2, are easy. When $M \equiv S^{\vec{P}} \xrightarrow{R}\!\!\!\!\rightarrow_1 T^{\vec{Q}} \equiv N$, use the facts that $M[x := A]$ is $S^{\overline{P[x:=A]}}$ and $N[x := B]$ is $T^{\overline{Q[x:=B]}}$, and that $P_i[x := A] \xrightarrow{R}\!\!\!\!\rightarrow_1 Q_i[x := A]$ by the induction hypothesis. □

The important feature of $\xrightarrow{R}\!\!\!\!\rightarrow_1$ is that we can project and develop a single step $\xrightarrow{R}\!\!\!\!\rightarrow_1$ reduction to a *single step* $\xrightarrow{R}\!\!\!\!\rightarrow_1$ reduction, as follows.

Proposition 4.6 *Let M be any $\Lambda(\Sigma)$ term. If $\rho : M \xrightarrow{\beta} X$ and $M \xrightarrow{R}\!\!\!\!\rightarrow_1 N$, then there are X' and Z such that $X \xrightarrow{\beta}\!\!\!\!\rightarrow X' \xrightarrow{R}\!\!\!\!\rightarrow_1 Z$ and $N \xrightarrow{\beta}\!\!\!\!\rightarrow Z$.*

Proof. By induction on the derivation of $M \xrightarrow{R}\!\!\!\!\rightarrow_1 N$.

1. $M \equiv N$: trivial.
2. $M \equiv (M_1M_2) \xrightarrow{R}\!\!\!\!\rightarrow_1 (N_1N_2) \equiv N$, with each $M_i \xrightarrow{R}\!\!\!\!\rightarrow_1 N_i$: There are two subcases.
   (a) If the redex term of ρ is a subterm of , (say), M_1, then $X \equiv (X_1X_2)$ with $M_1 \xrightarrow{\beta} X_1$. By induction, there exists X_1' and Z_1 with $X_1 \xrightarrow{\beta}\!\!\!\!\rightarrow X_1' \xrightarrow{R}\!\!\!\!\rightarrow_1 Z_1$ and $N_1 \xrightarrow{\beta}\!\!\!\!\rightarrow Z_1$. We can then take $Z \equiv (Z_1N_2)$.
   (b) If the redex term of ρ is M itself, then $M \equiv (\lambda xP)M_2, X \equiv P[x := M_2]$, with $\lambda xP \xrightarrow{R}\!\!\!\!\rightarrow_1 N_1$ and $M_2 \xrightarrow{R}\!\!\!\!\rightarrow_1 N_2$. Since $\xrightarrow{R}\!\!\!\!\rightarrow_1 \subseteq \xrightarrow{R}\!\!\!\!\rightarrow$, N_1 is of the form λxQ with $P \xrightarrow{R}\!\!\!\!\rightarrow_1 Q$. Then we can take Z to be $Q[x := N_2]$ and invoke Lemma 4.5.
3. $M \equiv \lambda xM_1 \xrightarrow{R}\!\!\!\!\rightarrow_1 \lambda xN_1 \equiv N$: similar to part 2a above.
4. $M \equiv S^{\vec{P}} \xrightarrow{R}\!\!\!\!\rightarrow_1 T^{\vec{Q}} \equiv N$, with each $P_i \xrightarrow{R}\!\!\!\!\rightarrow_1 Q_i$: If $(\lambda xA)B$ is the redex term in ρ then there is an i such that $(\lambda xA)B \sqsubseteq P_i$. For this i, we have a β-reduction out of P_i and $P_i \xrightarrow{R}\!\!\!\!\rightarrow_1 Q_i$, so by induction there are P_i' and Q_i', with $P_i \xrightarrow{\beta}\!\!\!\!\rightarrow P_i'$, $Q_i \xrightarrow{\beta}\!\!\!\!\rightarrow Q_i'$, and $P_i' \xrightarrow{R}\!\!\!\!\rightarrow_1 Q_i'$. Let $\vec{P}'$ denote the sequence of terms obtained from $\vec{P}$ by replacing P_i by

P'_i, and take X' to be $S^{\vec{P}'}$. Let $\vec{Q}'$ denote the sequence of terms obtained from $\vec{Q}$ by replacing Q_i by Q'_i, and take Z' to be $T^{\vec{Q}'}$. Then $X \equiv S^{\vec{P}} \xrightarrow{\beta}\!\!\!\!\rightarrow S^{\vec{P}'} \equiv X'$ by suitably reducing *all* occurrences of P_i to P'_i, and $X' \xrightarrow{R}\!\!\!\!\rightarrow_1 Z$ since each element of P' reduces via $\xrightarrow{R}\!\!\!\!\rightarrow_1$ to the corresponding element of Q'.

□

Preservation of confluence now follows.

Theorem 4.7 *If R is confluent on algebraic terms, then βR is confluent over R-stable $\Lambda(\Sigma)$ terms.*

Proof. Write $\beta nf(A)$ for the β-normal form of a term A. We first show, by induction along $\xrightarrow{\beta}$, that when M is stable and $M \xrightarrow{R}\!\!\!\!\rightarrow_1 N$ then $\beta nf(M) \xrightarrow{R}\!\!\!\!\rightarrow_1 \beta nf(N)$. If M is a β-normal form then by Lemma 2.10 .3 so is N. Otherwise, let $M \xrightarrow{\beta} X$ be any reduction, define X' and Z as in Proposition 4.6, and apply the induction hypothesis to the instance $X' \xrightarrow{R}\!\!\!\!\rightarrow_1 Z$.

It follows that M stable and $M \xrightarrow{R} N$ imply that $\beta nf(M) \xrightarrow{R}\!\!\!\!\rightarrow \beta nf(N)$.

Now, to show confluence, suppose M is R stable, with $A \xleftarrow{\beta R}\!\!\!\!\leftarrow M \xrightarrow{\beta R}\!\!\!\!\rightarrow B$. Then $\beta nf(A) \xleftarrow{R}\!\!\!\!\leftarrow \beta nf(M) \xrightarrow{R}\!\!\!\!\rightarrow \beta nf(B)$. Confluence of R on $\Lambda(\Sigma)$ yields P such that $\beta nf(A) \xrightarrow{R}\!\!\!\!\rightarrow P \xleftarrow{R}\!\!\!\!\leftarrow \beta nf(B)$, so that $A \xrightarrow{\beta R}\!\!\!\!\rightarrow P \xleftarrow{\beta R}\!\!\!\!\leftarrow B$ as desired. □

The proofs above suggest an approach to proving inheritence of confluence in (β-strongly normalizing) typed βR systems such as those using a polymorphic type discipline, (or a system of dependent types). The preservation of R-confluence on the set $\Lambda(\Sigma)$ should hold just as in Theorem 4.1 above. In these systems, reductions explicitly involving types are defined, so in such a system let $\xrightarrow{\beta+}$ stand for term β-reduction together with type reduction. The set of terms which type-check will be stable in the sense obtained by replacing β by $\beta+$ in Definition 1.1. When part 2 of the definition of $\xrightarrow{R}\!\!\!\!\rightarrow_1$ is expanded so that $\xrightarrow{R}\!\!\!\!\rightarrow_1$ is compatible with all of the term-forming operations, it suffices to prove a version of Proposition 4.6 in which β there is replaced by $\beta+$.

For example, in the Girard-Reynolds system $\lambda^{\forall}$ of polymorphic types, let $\xrightarrow{\beta'}$ be β-reduction on types. When algebraic rewriting is added, stability of type-checking terms (with respect to $\xrightarrow{\beta\beta'}$) is clear, and the proof of algebraic confluence on type checking terms is exactly as in Theorem 4.1. When the definiton of $\xrightarrow{R}\!\!\!\!\rightarrow_1$ is expanded so that $A \xrightarrow{R}\!\!\!\!\rightarrow_1 B$ implies $(\lambda t A) \xrightarrow{R}\!\!\!\!\rightarrow_1 (\lambda t B)$ and $(A\tau \xrightarrow{R}\!\!\!\!\rightarrow_1 B\tau)$, for type variables t and types τ, Proposition 4.6 holds when the relation $\xrightarrow{\beta}$ there is replaced by $\xrightarrow{\beta} \cup \xrightarrow{\beta'}$. Preservation of confluence then follows just as in Theorem 4.7.

Acknowledgements. I am indebted to Val Breazu-Tannen and Jean Gallier for illuminating discussions about the topics of this work, and to Patricia Johann for careful readings and suggestions for improvement. The referees helped to clarify the notion of stability and provided a substantial number of corrections.

References

[Bar84] H. P. Barendregt. *The Lambda Calculus: Its Syntax and Semantics.* North-Holland, Amsterdam, 1981, revised 1984.

[Bre87] V. Breazu-Tannen. *Conservative extensions of type theories.* dissertation, Massachusetts Institute of Technology, 1987.

[Bre88] V. Breazu-Tannen. Combining algebra and higher-order types, in *Proceedings of the Third Annual Symposium on Logic in Computer Science*, pp. 82- 90, 1988.

[BG89] V. Breazu-Tannen and J. Gallier. Polymorphic rewriting conserves algebraic strong normalization and confluence, in *Proceedings,* 16^{th} *ICALP*, Springer- Verlag, 1989.

[BM88] V. Breazu-Tannen and A. R. Meyer. Computable values can be classical, in *Proceedings of the Second Annual Symposium on Logic in Computer Science*, pp. 238-245, 1988.

[CH88] T. Coquand and G. Huet. The Calculus of Constructions, *Information and Control*, v.76, no.2/3, pp. 95-120, 1988.

[Der87] N. Dershowitz. Termination of rewriting, *J. Symbolic Computation* **3**, pp. 69-116, 1987.

[Gir71] J-Y. Girard. Une extension de l'interprétation de Gödel à l'analyse, et son application à l'elimination des coupures dans l'analyse et la théorie des types, in *Proc. Second Scandinavian Logic Symposium*, ed. J.E. Fenstad, North-Holland, Amsterdam, 1971.

[Gir72] J-Y. Girard. *Interprétation functionelle et élimination des coupures de l'arithmétique d'ordre supérieur*, These D'Etat, Universite Paris VII, 1972.

[Hue80] G. Huet. Confluent reductions: abstract properties and applications to term rewriting systems, *JACM* **27**, pp. 797-821, 1980.

[HL79] G. Huet, J.J. Lévy. Call by need computations in non-ambiguous linear term rewriting systems, Rapport Laboria 359, INRIA, 979.

[HO80] G. Huet, D. Oppen. Equations and rewrite rules: a survey, in Formal Languages: Perspectives and Open Problems, ed. R. Book, Academic Press, New York, 1980.

[Klo80] J. W. Klop. *Combinatory Reduction Systems*, Mathematical Center Tracts 127, Amsterdam, 1980.

[Mac86] D. B. MacQueen. Using dependent types to express modular structure, in *Conference Record of the Thirteenth Annual ACM Symposium on Principles of Programming Languages*, pp. 277-286, 1986.

[Mid89] A. Middeldorp. Modular aspects of properties of term rewriting systems related to normal forms, in *Proc. Third International Conference on Rewriting Techniques and Applications*, Springer-Verlag LNCS 355, pp. 263-277, 1989.

[Pot81] G. Pottinger. The Church-Rosser theorem for the typed λ calculus with surjective pairing, *Notre Dame Journal of Formal Logic*, v. 22, no. 3, pp. 264-268, 1981.

[Rey74] J. C. Reynolds. Towards a theory of type structure, in *Proc. Colloque sur la Programmation*, Springer-Verlag LNCS 19, pp. 408-425, 1974.

[Rus87] M. Rusinowitch. On termination of the direct sum of term rewriting systems, *Information Processing Letters* **26** pp.65-70, 1987.

[Toy87] Y. Toyama. On the Church-Rosser property for the direct sum of term rewriting systems, *Journal of the ACM*, v.34, no.1, pp.128- 143, 1987.

[TKB89] Y. Toyama, J. W. Klop and H. Barendregt. Termination for the direct sum of left-linear term rewriting systems, in *Proc. Third International Conference on Rewriting Techniques and Applications*, Springer-Verlag LNCS 355, pp. 477-491, 1989.