Practice Examination #3 Solutions

Note: This practice examination contains more problems than would typically be found on a real examination.

PROBLEM 1
Let a finite set \( S \) have \( n \) elements. How many different relations are there on \( S \)? Justify your answer.

Hint: Consider the entries of the possible relation matrices.

The relation matrix has \( N^2 \) entries, each of which can be 0 or 1. Therefore, there are \( 2^{N^2} \) possible relations.

PROBLEM 2
Part A
Let relation \( r \) on set \( A = \{a, b, c, d, e\} \) have the following adjacency matrix:

\[
R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

Draw the Hasse diagram for \( r \).

\[a \quad d \quad b \quad e \quad c\]

Part B
Find another relation \( s \) on \( A \) such that \( r \subset s \) and \( s \) also has a Hasse diagram. Draw the Hasse diagram for \( s \). Note: If there is more than one such \( s \), you need to draw only one.

\[a \quad a \quad d \quad d \quad b \quad e \quad e \quad b \quad c \quad c\]
PROBLEM 3
Draw the Hasse diagram for all partial orders on $S = \{a, b\}$. There are only a few.

\[
\begin{array}{ccc}
  a & b & a \\
  \| & \| \\
  b & a
\end{array}
\]

PROBLEM 4
Let function $f: A \to A$ be a symmetric relation.

Part A
Prove that $f^{-1} = f$.

Let $f(a) = b$. There are 2 cases to consider: $a = b$ and $a \neq b$. If $a = b$, then $f(a) = a$, and $f(f(a)) = f(a) = a$. If $a \neq b$, then by symmetry, $f(b) = a$ and $f(f(a)) = f(b) = a$. In either case, $f \circ f = id$ and so $f = f^{-1}$.

Part B
Must $f$ be reflexive? Why or why not?

No. consider $f(n) = 1 - n$ on the set $\{0, 1\}$. It is not reflexive, yet it satisfies $f^{-1} = f$.

PROBLEM 5
Let $[V, E]$ be an undirected simple graph with vertices $V$ and edges $E$. Define relation $r$ on $V$ by $v_1rv_2$ iff there is a path (sequence of edges) connecting $v_1$ to $v_2$. Assume that every vertex is connected to itself. Prove that $r$ is an equivalence relation on $V$.

Hint: What 3 things must hold for an equivalence relation?

Reflexive: It is assumed that every vertex connects to itself.

Symmetric: If $v_1rv_2$ then there is a sequence of edges $e_1, e_2, \ldots, e_j$ from $v_1$ to $v_2$. Because we are dealing with undirected graphs, the sequence of edges $e_j, \ldots, e_2, e_1$ connects $v_2$ to $v_1$. Therefore $v_2rv_1$ and the relation is symmetric.

Transitive: If $v_1rv_2$ and $v_2rv_3$ then $v_1rv_3$ simply by putting together the edge sequences (paths) from $v_1$ to $v_2$ and from $v_2$ to $v_3$.

PROBLEM 6
A relation $R$ on set $S$ is anti-transitive if $\forall a, b, c, \in S, (aRb) \land (bRc) \rightarrow \neg(aRc)$.

Part A
Prove that if relation $R$ is anti-transitive, then $R$ is anti-reflexive. Hint: Try an indirect proof.

Suppose that $R$ is not anti-reflexive. Then there exists $a \in S$ such that $aRa$. If $R$ were anti-transitive, then $(aRa) \land (aRa) \rightarrow \neg(aRa)$. This cannot be, so the relation cannot be anti-transitive. Therefore, if $R$ is anti-transitive, it must be anti-reflexive.
Part B
Find an anti-transitive relation. You may use a graph, relation matrix or other *unambiguous*
means to describe the relation. Be sure to specify the set.

Let the set be \( \{a, b, c\} \) with relation \( R = \{(a, b), (b, c)\} \).

**PROBLEM 7**
In this problem, a partition is considered to be a set of subsets. Consequently, the following
partitions of set \( \{a, b, c\} \) are considered to be the same partition:

\[
\{\{a\}, \{b, c\}\}, \ \{\{a\}, \{c, b\}\}, \ \{\{b, c\}, \{a\}\}, \ \{\{c, b\}, \{a\}\}
\]

How many distinct ways can a set with \( N \) elements be partitioned into \( N-1 \) *non-empty* subsets?
Explain.

There must be \( N - 2 \) subsets with a single element and 1 subset with 2 elements. We need only
pick 2 elements from the set—this defines the doubleton subset and the other subsets are the
singletons. We can pick the 2 elements in \( \binom{N}{2} \) ways.