Either prove or give a counterexample to establish whether or not each of the following conjectures holds for every connected weighted graph \( G = (V, E) \), \( w : E \to \mathbb{R}^+ \), with \( |E| > |V| \geq 3 \), with distinct edge weights? You may use the fact that if the edge weights are distinct then \( G \) admits exactly one MST.

**Conjecture 1:** If \( e \) is the second lightest edge of \( E \), then \( e \) belongs to every minimum spanning tree of \( G \).

The conjecture is true. Let \( uv \) be a lightest edge. Applying the Blue Rule in cut \( \{u\} \cup (V - \{u\}) \) colors \( uv \) Blue. Letting \( A \) be the set containing \( \{u, v\} \) plus one endpoint of the second lightest edge, an application of the Blue Rule will color the second lightest edge Blue, assuring us that it belongs to an MST.

**Conjecture 2:** If \( e \) is the third lightest edge of \( E \), then \( e \) belongs to every minimum spanning tree of \( G \).

Edge \( uv \) in the following graph is the third lightest edge, but it does not belong to an MST.

**Conjecture 3:** If \( e \) is the heaviest edge of \( E \), then \( e \) does not belong to any minimum spanning tree of \( G \).

Edge \( uv \) in the following graph is the heaviest edge, but it belongs to the MST.