1 Exercise 34.1-1 on page 1060 of our text.

**SOLUTION.** If LONGEST-PATH-LENGTH could be solved in polynomial time, then we could solve the decision problem LONGEST-PATH in polynomial time for instance \( \langle G, u, v, k \rangle \) by running LONGEST-PATH-LENGTH on \( \langle G, u, v \rangle \) and returning true if and only if LONGEST-PATH-LENGTH returned a value \( \geq k \).

If LONGEST-PATH \( \in P \), then we identify the length of the longest path by essentially doing a sequential search (binary search would be faster, though still in \( P \)) over all possible path lengths.

\[
\text{LONGEST-PATH}(G,u,v) \\
\quad k \leftarrow 0 \\
\quad \text{while } \text{LONGEST-PATH}(G,u,v,k+1) \text{ do } k \leftarrow k + 1 \\
\quad \text{return } k
\]

2 Given a graph \( G = (V, E) \), a set of vertices \( U \subseteq V \) is an independent set if no pair of vertices of \( U \) has an edge between them. Consider the INDEPENDENT SET problem:

**INSTANCE:** Graph \( G = (V, E) \) and \( k \in \mathbb{Z}^+ \).

**QUESTION:** Does \( G \) have an independent set of cardinality \( k \)?

**a** Prove that the INDEPENDENT SET problem is NP-complete.

**Hint:** You may want to use the VERTEX-COVER problem described in Section 34.5.2 of our text.

**b** Suppose you have an Oracle which will decide on the INDEPENDENT SET problem in constant time. That is, the Oracle will accept as input a graph \( G = (V, E) \) and \( k \in \mathbb{Z}^+ \) and will tell, in constant time, whether or not \( G \) has an independent set of cardinality \( k \). Show how to use the Oracle to determine the largest independent set of \( G \) in polynomial (in \( |V| \) and \( |E| \)) time. Note that we seek an answer to the optimization problem using a solution to the decision problem (the Oracle). Analyze your algorithm.

**SOLUTION:** a For any given certificate \( U \subseteq V \), it is easy to verify in polynomial time, \( O\left(\binom{|U|}{2}\right) = O(n^2) \) time, that \( U \) is an independent set. Thus, the INDEPENDENT SET problem belongs to NP.

It is fairly easy to see that \( U \subseteq V \) is an independent set of \( G \) if and only if its complement, \( V \setminus U \), is a vertex cover. Since the text shows that the VERTEX-COVER problem is NP-complete, we only need show that an instance of VERTEX-COVER is polynomially reducible to an instance of INDEPENDENT SET. Since \( G \) doesn’t even change, this is trivially true. If there were a polynomial time algorithm to solve INDEPENDENT SET, then we could solve VERTEX-COVER in polynomial time by responding that \( G \) admits a Vertex-Cover of cardinality \( k \) if and only if it admits an independent set of cardinality \( |V| - k \).
First, we compute the cardinality of the maximum independent set of $G$.

\begin{align*}
  k & \leftarrow 0 \\
  \textbf{while } \text{Oracle}(G,k+1) \text{ do } & k \leftarrow k + 1
\end{align*}

Now $k$ contains the cardinality of the maximum independent set of $G$. For any $v \in V$ we let $G/v$ denote graph $G$ with vertex $v$ removed, as well as all edges incident with $v$. We now use Oracle to find an independent set of cardinality $k$.

\begin{align*}
  \textbf{for each } v \in V \text{ if } \text{Oracle}(G/v,k) & \quad \Rightarrow G/v \text{ still contains an independent set of } k \text{ vertices} \\
  \text{then } G & \leftarrow G/v
\end{align*}

the vertices of $G$ are now an independent set of $k$ vertices

Exercise 34.2-1 on page 1065 of our text.

\textbf{SOLUTION:} A certificate for isomorphism of $G = (V, E)$ and $G' = (V', E')$ is a bijection $f : V \rightarrow V'$. The fact that $f$ preserves adjacency can be verified be checking adjacency of the $\binom{|V'|}{2} \in O(|V'|^2)$ pairs of vertices. That is, \textbf{for each } $u, v \in V$ we verify that $(u, v) \in E$ if and only if $(f(u), f(v)) \in E'$.

Exercise 34.5-7 on page 1101 of our text.
**SOLUTION:** We know that the HAMILTON CYCLE problem is NP-complete. It is easy to see that the HAMILTON-CYCLE problem ≤ₚ the LONGEST-SIMPLE-CYCLE problem because graph $G = (V, E)$ admits a Hamilton cycle if and only if the length of its longest simple cycle equals $|V|$.

5 ▶ The problem of deciding if you can divide $n$ integers $\{p_1, \ldots, p_n\}$ into two sets with equal sums is NP-complete. Show that the KNAPSACK PROBLEM is NP-complete.

**SOLUTION:** We convert any instance $\{p_1, \ldots, p_n\}$ of our problem (the PARTITION PROBLEM) into an instance of the KNAPSACK PROBLEM in which object $i$, $1 \leq i \leq n$, has value and weight $p_i$. The capacity of the knapsack is $\frac{\sum_{1 \leq i \leq n} p_i}{2}$, and the instance of PARTITION PROBLEM admits a solution if and only if the KNAPSACK PROBLEM admits a solution of value $\frac{\sum_{1 \leq i \leq n} p_i}{2}$. 