Name________________________

Date: December 17, 2009
All documentation permitted, but no communication is permitted

1. (20 points) Which of the following are context-free? Justify your answers.
   \[ a \{a^n b^n c^{3n} \mid n \geq 0\} \]

   \[ b \{a^n b^m c^{3n} \mid m, n \geq 0\} \]
2. (30 points) Let \( \mathcal{P} \) be the set of languages for which membership can be decided by a Turing machine in polynomial time, and let \( \mathcal{NP} \) be the set of languages for which membership can be decided by a nondeterministic Turing machine in polynomial time.

\( a \) Is \( \mathcal{P} \) closed under concatenation? Justify your answer.

\( b \) Is \( \mathcal{NP} \) closed under concatenation? Justify your answer.

\( c \) Is \( \mathcal{P} \) closed under complement? Justify your answer.
3. (20 points) Is the following CONJECTURE true? Justify your answer.

**CONJECTURE**: The set of recursively enumerable languages is closed under complement.
4. (30 points) Consider the language

\[ L = \left\{ (M_0, M_1, z) \mid \text{TM } M_0 \text{ halts on input } z \in \Sigma^* \text{ in fewer steps than TM } M_1 \right\}. \]

Note that if \( M_0 \) halts on \( z \) but \( M_1 \) does not, then \( (M_0, M_1, z) \in L \).

\( \text{a} \) Is \( L \) recursively enumerable? Justify your answer.

\( \text{b} \) Is \( L \) recursive? Justify your answer.
1. $a \{a^n b^n c^{3n} \mid n \geq 0 \}$ is not a CFL. Assume it is a CFL, and let $k$ be the constant provided by the Pumping Lemma for CFLs. Consider $z=a^k b^k c^{3k}$. By the PL, $z$ can be written $uvwxy$ with $\text{length}(v) + \text{length}(x) > 0$ and $\text{length}(vwx) \leq k$ such that $\{uv^iwx^i \mid i \geq 0\} \subseteq \{a^n b^n c^{3n} \mid n \geq 0\}$.

Because $\text{length}(vwx) \leq k$, $vwx$ cannot contain $a$'s and $c$'s. But because $\text{length}(v) + \text{length}(x) > 0$, string $vx$ must contain some $a$'s or $b$'s or $c$'s. So $uv^2wx^2y$ cannot increase the number of $a$'s, $b$'s and $c$'s. So $uv^2wx^2y \not\in \{a^n b^n c^{3n} \mid n \geq 0\}$, which is a contradiction.

2. $a \cup b \in \mathcal{P}$ and $\mathcal{NP}$ are each closed under concatenation. Assume that $L_0, L_1 \in \mathcal{P}$ (respectively $L_0, L_1 \in \mathcal{NP}$). There must exist TMs $M_0, M_1$ (respectively TMs $M_0, M_1$) such that $L_0 = L(M_0)$ and $L_1 = L(M_1)$, and $M_0$ and $M_1$ decide on membership in $L_0$ and $L_1$ in polynomial time. We want to show that $L_0 \cdot L_1 \in \mathcal{P}$ (respectively $L_0 \cdot L_1 \in \mathcal{NP}$).

For both cases (deterministic and nondeterministic), we can test whether $z \in L_0 \cdot L_1$ by using the following algorithm:

```plaintext
for each $x, y \in \Sigma^*$ such that $xy = z$ do
    if $x \in L(M_0)$ and $y \in L(M_1)$ then return accept
return reject
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There are $\text{length}(z) + 1$ decompositions of $z$ into $x$ and $y$, so there are a polynomial (in $\text{length}(z)$) number of invocations of two polynomial time algorithms. So the test is performed in polynomial time. And the algorithm decides that $z \in L_0 \cdot L_1$ if and only if $z$ can be decomposed into $x$ and $y$ such that $x \in L_0$ and $y \in L_1$.

3. $c \in \mathcal{P}$ is closed under complement. Assume that $L \in \mathcal{P}$. There must exist TM $M$ such that $L = L(M)$ and $M$ decides on membership in $L$ in polynomial time. To decide whether $z \in \overline{L}$, we run the following TM which halts in polynomial time.

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run $M$ on input $z$
    if $M$ returns accept then return reject else return accept
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3. The set $HALT$ of pairs $(M, z)$ where TM $M$ halts on input $z$ is recursive but not recursive (this is equivalent to the halting problem). We proved in class, and it is proved in the text, that
If $L$ and $\overline{L}$ are both re then $L$ is recursive. So $HALT$ is re, but its complement is not re. So the CONJECTURE is false.

4. **a** $L$ is re. We can use two Universal TMs to simulate, step by step, $M_0$ and $M_1$ on input $z$. If $M_0$ halts first, then we accept. If $M_1$ halts first, then we reject.
   **b** $L$ is not recursive. The HALTING PROBLEM is reducible to membership in $L$, and hence membership in $L$ can not be recursive. Let $M_\infty$ be a TM which never halts. For example, $M_\infty$ can have one state, $q_0$, and $\delta_{M_\infty}(q_0,x) = [q_0,x,R]$ for all $x \in \Gamma$. Given any instance of the HALTING PROBLEM, such as whether TM $M$ halts on input $z$, we note that the answer is yes if and only if $(M,M_\infty,z) \in L$. Hence, if membership in $L$ were recursive, then the HALTING PROBLEM would be decidable.