1. (1 point) Do Exercise 27 on page 38 of the text.

2. (16 points) Prove Theorem 1.4.4 on pages 18 and 19 of our text.

3. (10 points) Assume that $X$ and $Y$ are infinite disjoint sets of nodes and consider the digraph with nodes $N = X \cup Y$. In the digraph, each node of $X$ (respectively $Y$) has exactly one arc going out (it has out-degree 1) and it goes to a node of $Y$ (respectively $X$). Furthermore each node has in-degree at most 1. A component of a digraph is a minimal set of nodes (and the arcs between them) such that every pair of nodes with a path between them belong to the same component.

   i) Describe the possible components of the digraph.
   
   ii) For each possible component, describe a bijection between the nodes from $X$ in the component and the nodes from $Y$.

   iii) Conclude Theorem 1.4.2 for infinite disjoint sets.

4. (5 points) Give regular expressions for the following languages over $\Sigma = \{0,1\}$:

   i) Strings of length less than 4.
   
   ii) Strings that contain 001 as a substring.
   
   iii) Strings with no pair of consecutive 1’s.
   
   iv) Strings whose next to last character is a 1.
   
   v) Strings that do not begin with 00.

5. (2 points) Give a string $u \in \{0,1\}^*$ such that $\text{length}(u) = 8$ but $u$ does not belong to the regular language $(00 \cup 11)^* (01 \cup 10)(00 \cup 11)^*$.

6. (2 points) Give a regular expression for the language of assignment statements such as $E \leftarrow 2.71828$ or $PI \leftarrow 3.14159265358979323$ An assignment statement is a nonempty string of uppercase letters, followed by a left arrow, "$\leftarrow$", followed by a nonempty string of digits without a leading 0, followed by a decimal point, ".", followed by a nonempty string of digits without a trailing 0.
1. If \( \text{card}(X) \equiv \text{card}(X) \), then there is a bijection \( f : X \rightarrow X \). The inverse \( f^{-1} \) is also a bijection, so \( \equiv \) is reflexive.

   If \( \text{card}(X) \equiv \text{card}(Y) \), then there is a bijection \( f : X \rightarrow Y \). The inverse \( f^{-1} : Y \rightarrow X \) is also a bijection, so \( \equiv \) is symmetric.

   If \( \text{card}(X) \equiv \text{card}(Y) \) and \( \text{card}(Y) \equiv \text{card}(Z) \), then there are bijections \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \). The functional composition \( gf : X \rightarrow Z \) a bijection, so \( \equiv \) is transitive.

2. i) Let \( X \) and \( Y \) be countable sets. Set \( Y^* = Y - X \).
   - If \( X \) and \( Y^* \) are both finite, \( X \cup Y = X \cup Y^* \) is finite and countable.
   - If one of \( X \) and \( Y^* \) are finite (without loss of generality assume \( X \) is finite, and that \( X = \{x_0, x_1, \ldots, x_n\} \) and \( Y^* = \{y_0, y_1, y_2, \ldots\} \), then
     \[
     \begin{array}{cccccccc}
     0 & 1 & 2 & \ldots & n & n+1 & n+2 & \\
     \downarrow & \downarrow & \downarrow & \ldots & \uparrow & \uparrow & \uparrow & \\
     x_0 & x_1 & x_2 & x_n & y_0 & y_1 & y_2 & \\
     \end{array}
     \]
     demonstrates the countability of \( X \cup Y = X \cup Y^* \).
   - If \( X \) and \( Y^* \) are both infinite, then
     \[
     \begin{array}{cccccccc}
     0 & 1 & 2 & 3 & 4 & 5 & \ldots \\
     \downarrow & \downarrow & \downarrow & \downarrow & \uparrow & \uparrow & \uparrow & \\
     x_0 & y_0 & x_1 & y_1 & x_2 & y_2 & \\
     \end{array}
     \]
     demonstrates the countability of \( X \cup Y = X \cup Y^* \).

ii) We use the dovetailing construction of Example 1.4.2 with \( X \) listed along the x-axis and \( Y \) listed along the y-axis. If either \( X \) or \( Y \) is finite, we remove the ordered pairs corresponding to the nonexistent elements from the enumeration.

iii) Let \( X = \{x_0, \ldots, x_{n-1}\} \) or \( X = \{x_0, x_1, \ldots\} \). We know that \( \mathbb{N} \) is countably infinite, and we list it, expressing each number in binary in reverse order of the bits:

\[
\begin{align*}
0 & \leftrightarrow 0 \\
1 & \leftrightarrow 1 \\
2 & \leftrightarrow 01 \\
3 & \leftrightarrow 11 \\
4 & \leftrightarrow 001 \\
5 & \leftrightarrow 101 \\
\ldots & \notag
\end{align*}
\]
We use each binary string $b_0b_1b_2 \ldots b_k$ to represent the subset $\{x_i | x_i = 1\}$. That is, string 00101 represents subset $\{x_2, x_4\}$. So the finite subsets of $X$ are

$$
\begin{align*}
0 &\leftrightarrow \emptyset \\
1 &\leftrightarrow \{x_0\} \\
2 &\leftrightarrow \{x_1\} \\
3 &\leftrightarrow \{x_0, x_1\} \\
4 &\leftrightarrow \{x_2\} \\
5 &\leftrightarrow \{x_0, x_2\}
\end{align*}
$$

iv) Let $X = \{x_0, \ldots, x_{n-1}\}$ or $X = \{x_0, x_i, \ldots\}$, and let $Y$ be the set of finite length sequences of elements of $X$. Since $\{\lambda, x_0, x_0x_0, x_0x_0x_0, \ldots\} \subseteq Y$, it follows that $Y$ is infinite. The set of all sequences of length 0 are countable (there is only sequence $\lambda$), as well as the set of all sequences of length 1 (this is $X$, which is countable), as well as the set of all sequences of length 2 (using the dovetailing technique of Example 1.4.2), as well as the set of all sequences of length 3 (using the dovetailing technique of Example 1.4.2 on sequences of lengths 2 and 1),... Let us denote these sequences as $\sigma_0, \sigma_1, \sigma_2, \ldots$ We finally use the construction of Example 1.4.2 to dovetail these sequences together:

$\sigma_0 \lambda$

$\sigma_1 x_0 x_1 \ldots x_{n-1}$

$\sigma_2 x_0x_0 x_0x_1 \ldots x_0x_{n-1} x_1 x_0 x_1 \ldots x_1 x_{n-1} \ldots x_0 x_{n-1} x_1 \ldots x_{n-1} x_{n-1}$

$\sigma_3 \ldots$

3. i) The components can be one-way infinite paths, two-way infinite paths and even length cycles. All the paths and cycles alternate between nodes of $X$ and nodes of $Y$. Examples of these three types of components are:

- One-way infinite path
  $X$ is the even natural numbers and $Y$ is the odd natural numbers and the arcs are $\{(n, n+1) | n \in \mathbb{N}\}$.

- Two-way infinite path
  $X$ is the even integers and $Y$ is the odd integers and the arcs are $\{(n, n+1) | n \in \mathbb{Z}\}$.

- Cycle
  $X = \{0\}$ and $Y = \{1\}$ and the arcs are $\{[0,1],[1,0]\}$.

ii) If the component is a one-way infinite path, then the bijection contains the arc from the first node of the path and alternates arcs thereafter.
If the component is a two-way infinite path, the bijection contains the arcs from the nodes of $X$.

If the component is a cycle, the bijection contains the arcs from the nodes of $X$.

**iii)** $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \leq \text{card}(X)$ means that there exist total one-to-one functions from $X$ into $Y$ and from $Y$ into $X$. These functions yield the arcs of the digraph. The union of the bijections from the different components yields a bijection between $X$ and $Y$.

4. i) $\lambda \cup 0 \cup 1 \cup 00 \cup 01 \cup 10 \cup 11 \cup 000 \cup 001 \cup 010 \cup 011 \cup 100 \cup 101 \cup 110 \cup 111$

ii) $(0 \cup 1)^* 001 (0 \cup 1)^*$

iii) Every 1 must either be followed by a 0 or be the last character of the string.

$$(0 \cup 10)^* (\lambda \cup 1)$$

or, if you don’t want to use $\lambda$ in the regular expression, $(0 \cup 10)^* \cup (0 \cup 10)^* 1$.

iv) $(0 \cup 1)^* 1 (0 \cup 1)$

v) $\lambda \cup 0 \cup 1 (0 \cup 1)^* \cup 01 (0 \cup 1)^*$

5. $u=00000000$

6. $(A \cup \ldots \cup Z) (A \cup \ldots \cup Z)^* \leftarrow (1 \cup \ldots \cup 9) (0 \cup \ldots \cup 9)^* (0 \cup \ldots \cup 9)^* (1 \cup \ldots \cup 9)$