1. (8 points) Which one of the following two sets is r.e.? Justify your answer by giving a proof that it is r.e.

\{ M \mid L(M) \text{ contains at least 1024 elements} \}
\{ M \mid L(M) \text{ contains at most 1024 elements} \}

2. (10 points) Is the following question decidable?

**INPUT:** Nfas \( N_1 = (Q_1, \Sigma, \Delta_1, S_1, F_1) \), \( N_2 = (Q_2, \Sigma, \Delta_2, S_2, F_2) \), \( N_3 = (Q_3, \Sigma, \Delta_3, S_3, F_3) \).

**QUESTION:** Is \( L(N_1) \cap L(N_2) \cap L(N_3) \) infinite?

3. (10 points) Is the set of Turing machines
\( L_{1024} = \{ M \mid M \text{ halts within 1024 steps on all inputs} \} \) recursive? Justify your answer.
1. \( \{M \mid L(M) \text{ contains at least 1024 elements}\} \) is r.e. To enumerate it, we dovetail over all the machines

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad t \text{ (steps of simulation)} \\
M_0 & \quad 1 & \quad 2 & \quad 4 & \quad 7 & \quad 11 \\
M_1 & \quad 3 & \quad 5 & \quad 8 & \quad 12 \\
M_2 & \quad 6 & \quad 9 & \quad 13 \\
M_3 & \quad 10 & \quad 14
\end{align*}
\]

For the \( t \) steps of the simulation of \( M_i \), we dovetail over the inputs

\[
\begin{align*}
1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad t \text{ (steps of simulation)} \\
y_0 & \quad 1 & \quad 2 & \quad 4 & \quad 7 & \quad 11 \\
y_1 & \quad 3 & \quad 5 & \quad 8 & \quad 12 \\
\Sigma^* & \quad 6 & \quad 9 & \quad 13 \\
y_3 & \quad 10 & \quad 14
\end{align*}
\]

If we determine that \( M_i \) accepts 1024 inputs, we list it.

2. The question is decidable. Because \( L(M_1), L(M_2), L(M_3) \) are regular, and because regular languages are closed under intersection and concatenation, there is a DFA \( M \) to accept \( (L(N_1) \cap L(N_2))L(N_3) \). The language is infinite if and only if \( M \) has a state \( q \) such that:

- there is a path from the start state of \( M \) to \( q \), and
- there is a path from \( q \) to a final state of \( M \), and
- there is a nontrivial cycle from \( q \) to \( q \).

Because all three of these conditions are decidable for any state of any DFA, the question is decidable.

3. \( L_{1024} \) is recursive. If an \( M_i \) halts on any input within 1024 steps, then it can't get past the 1025th tape cell. There are only \( |\Sigma|^{1024} \) distinct possibilities, \( \{\mid \} \times \Sigma^{1024} \), for these first 1025 squares, So \( M_i \in L_{1024} \) if and only if it halts on every one of these \( |\Sigma|^{1024} \) different tapes.