DUE: Friday, September 30

1. (16 points) For each of the following languages, tell whether or not it is context-free and justify your answers.
   a \( \{a^i b^j c^k \mid 0 \leq i \leq j \leq k \} \).
   b \( \{a^i b^j c^k \mid i \neq j \lor j = k \} \).

2. (10 points) Prove that if \( L_0 \) is a regular language and \( L_1 \) is a CFL, then \( L_0 \cap L_1 \) must be a CFL.
1. $L = \{a^i b^j c^k | 0 \leq i \leq j \leq k\}$ is not a CFL. We prove this by assuming that it is context-free and deriving a contradiction from the Pumping Lemma for CFLs. Let $n$ be as provided by the PL for CFLs, and choose $z = a^n b^n c^n$. There must exist a decomposition $z = uvwx$ guaranteed by the PL. There are two cases to consider:

- If either $v$ or $x$ contains 2 types of symbols, then $uv^2wx^2y$ contains $a$'s, $b$'s and $c$'s in the wrong order. So $uv^2wx^2y \notin L$, although the PL says it must. So this is a contradiction.

- If each of $v$ and $x$ contains at most 1 type of symbol, there are three subcases:
  - If $vx$ doesn’t have any $a$’s, then $uvy$ has more $a$’s than $b$’s or more $a$’s than $c$’s. So $uv^0wx^0y = uvw \notin L$, although the PL says it must. So this is a contradiction.
  - If $vx$ doesn’t have any $b$’s, then if $vx$ has $a$’s, then $uv^2w^2y$ has more $a$’s than $b$’s. If $vx$ has $c$’s, then $uv^0wx^0y = uvw$ has more $b$’s than $c$’s. In either case, we get a contradiction from the PL.
  - If $vx$ doesn’t have any $c$’s, then $uv^2wx^2y$ either has more $a$’s than $c$’s or more $b$’s than $c$’s. In either case, we get a contradiction from the PL.

$b \{a^i b^j c^k | i \neq j \lor j = k\}$ is a CFL because it is the union of 2 CFLs, $\{a^i b^j c^k | i \neq j\}$ and $\{a^i b^j c^k | j = k\}$, and we know that CFLs are closed under union. It is easy to see that $\{a^i b^j c^k | i \neq j\}$ is generated by the CFG:

\[
\begin{align*}
S & \rightarrow S_{i,j} \mid S_{i<j} \\
S_{i>j} & \rightarrow AS_{i=j} C \\
A & \rightarrow aA \mid a \\
S_{i=j} & \rightarrow aS_{i=j} b \mid \varepsilon \\
C & \rightarrow cC \mid \varepsilon \\
S_{i<j} & \rightarrow S_{i=j} BC \\
B & \rightarrow bB \mid b
\end{align*}
\]

and that $\{a^i b^j c^k | j = k\}$ is generated by the CFG:

\[
\begin{align*}
S & \rightarrow AS' \\
A & \rightarrow aA \mid \varepsilon \\
S' & \rightarrow bS' c \mid \varepsilon
\end{align*}
\]
2. If $L_0$ is a regular language and $L_1$ is a CFL, then there are DFA $M_0 = (Q_0, \Sigma, \delta, s_0, F_0)$ and PDA $M_1 = (Q_1, \Sigma, \Gamma, \Delta, s_1, \perp, F_1)$ such that $L_0 = L(M_0)$ and $L_1 = L(M_1)$. We accept $L_0 \cap L_1$ by constructing a PDA $M^* = (Q_0 \times Q_1, \Sigma, \Gamma, \Delta^*, (s_0, s_1), \perp, F_0 \times F_1)$ which “simulates” $M_0$ and $M_1$ in parallel. That is, for each $q_k \in Q_0, q_i \in Q_1, a \in \Sigma, A \in \Gamma$,

$$\Delta^*\left((q_k, q_i), a, A\right) = \left\{ \left( (q_j, q_j), B_1 B_2 \ldots B_k \right) | \delta(q_k, a) = q_i \wedge \left( (q_j, a, A), (q_j B_1 B_2 \ldots B_k) \in \Delta \right) \right\}.$$