1. (6 points) Prove or give a counterexample to the following CONJECTURES.

**CONJECTURE 1:** For any NFA \( N = (Q, \Sigma, \Delta, S, F) \) and any \( q \in Q \), the language

\[
L_{N, \rightarrow q} = \left\{ w \in \Sigma^* | (\exists s \in S) q \in \hat{\Delta}(s, w) \right\}
\]

must be regular.

**CONJECTURE 2:** For any NFA \( N = (Q, \Sigma, \Delta, S, F) \) and any \( q \in Q \), the language

\[
L_{N, q \rightarrow} = \left\{ w \in \Sigma^* | (\exists q^* \in F) q^* \in \hat{\Delta}(q, w) \right\}
\]

must be regular.

2. (14 points) The operations \( \circ \) and \( \otimes \) over languages are defined by

\[
L_0 \circ L_1 = \left\{ w | (\exists x \in L_1) wx \in L_0 \right\}
\]

\[
L_0 \otimes L_1 = \left\{ wx | (\exists x \in L_1) w \in L_0 \right\}.
\]

Prove or give a counterexample to the following CONJECTURES. Hint: You may want to invoke your answers to Problem 1 above.

**CONJECTURE 1:** The set of regular languages is closed under the operation \( \circ \).

**CONJECTURE 2:** The set of regular languages is closed under the operation \( \otimes \).

3. (10 points) \( a \) Is \( \left\{ a^i b^j \mid i \geq 1 \right\} \cup \left\{ a^i b^j \mid i, j \geq 0 \right\} \) regular? Justify your answer.

\( b \) Is \( \left\{ a^i b^j c^j \mid i \geq 1 \land j \geq 0 \right\} \cup \left\{ a^i b^j \mid i, j \geq 0 \right\} \) regular? Justify your answer.
1. **a** The **Conjecture** is true because \( L_{N \rightarrow q} \) is accepted by the NFA \( N = (Q, \Sigma, \Delta, S, \{q\}) \).

**b** The **Conjecture** is true because \( L_{N,q} \) is accepted by the NFA \( N = (Q, \Sigma, \Delta, \{q\}, F) \).

2. Both **Conjectures** are true.

**Conjecture 1:** If \( L_0 \) and \( L_1 \) are regular, there must exist DFAs \( M_0 = (Q_0, \Sigma, \delta_0, s_0, F_0) \) and \( M_1 = (Q_1, \Sigma, \delta_1, s_1, F_1) \) such that \( L_0 = L(M_0) \) and \( L_1 = L(M_1) \). From Problem 1 we know that \( L_{M_0,q} \) is regular for each \( q \in Q_0 \). And because regular languages are closed under intersection, then \( L_{M_0,q} \cap L_1 \) must be regular. Because it is decidable whether a regular language is empty, we can decide for each \( q \in Q_0 \) whether \( L_{M_0,q} \cap L_1 \) is empty. So \( L_0 \cap L_1 \) is accepted by \( M^* = (Q_0, \Sigma, \delta_0, s_0, F^*) \) where \( F^* = \{ q \in Q_0 \mid L_{M_0,q} \cap L_1 \neq \emptyset \} \).

That is, we accept \( w \in \Sigma^* \) if there exists a string \( x \in \Sigma^* \) such that \( \delta_1(s_1, x) \in F_1 \) and \( \delta_0(s_0, w), x) \in F_0 \).

**Conjecture 2:** \( L_0 \otimes L_1 = L_0 L_1 \) and we know that the regular languages are closed under concatenation. So they are closed under \( \otimes \).

3. **a** \( \{a^i b^j | i \geq 1\} \cup \{a^i b^j | i, j \geq 0\} \), and it is regular because it is accepted by the DFA \( M = (\{q_a, q_b, q_{blackhole}\}, \{a, b\}, \delta, q_a, \{q_a, q_b\}) \) where

\[
\delta = \\
\begin{array}{ccc}
q_a & a & b \\
q_a & q_a & q_b \\
q_b & q_{blackhole} & q_b \\
q_{blackhole} & q_{blackhole} & q_{blackhole}
\end{array}
\]

**b** \( \{a^i b^j c^j | i \geq 1 \land j \geq 0\} \cup \{a^i b^j | i, j \geq 0\} \) is not regular. Let \( k \) be as specified by the Pumping Lemma for Regular Languages. We note that \( a^k b^k c \in \{a^i b^j c^j | i \geq 1 \land j \geq 0\} \cup \{a^i b^j | i, j \geq 0\} \) could be written as \( uvw \) with \( 1 \leq |v| \leq k \) and \( |uv| \leq k \) and \( \{uv^i w | l \geq 0\} \subseteq \{a^i b^j c^j | i \geq 1 \land j \geq 0\} \cup \{b^i c^j | i, j \geq 0\} \). But \( v \) must be a nonempty string of \( a \)'s. So \( uv^2 w \) has more \( a \)'s than \( b \)'s, and it can't belong to \( \{a^i b^j c^j | i \geq 1 \land j \geq 0\} \). And because it doesn't contain a \( c \) it can't belong to \( \{a^i b^j | i, j \geq 0\} \). So, by contradiction, \( \{a^i b^j c^j | i \geq 1 \land j \geq 0\} \cup \{a^i b^j | i, j \geq 0\} \) is not regular.