DUE: Monday, September 15

1. (10 points) We say that a state $q \in Q$ of an NFA $(Q, \Sigma, \Delta, S, F)$ is a blue state if there exists a string $z \in \Sigma^*$ such that $z=uv$ and $q \in \hat{\Delta}(S,u)$ and $\hat{\Delta}(\{q\},v) \cap F \neq \emptyset$. In other words, string $z$ passes from some initial state through $q$ on the way to a final state. It is possible for $N$ to have $\varepsilon$-transitions.

Give an algorithm to identify all blue states of an NFA. Note that an algorithm must terminate in finite time (it can not test all strings of $\Sigma^*$).

2. (8 points) Give a regular expression for each of the following languages over $\{0,1\}$.
   a Every 0 is either immediately preceded by a 1 or immediately followed by a 1.
   b The set of strings which do not contain a 00.

3. (3 points) Are the languages denoted in 2 a and 2 b equal to each other? Justify your answer.

4. (5 points) Which of the following "identities" are true for all regular expressions $\alpha$ and $\beta$? If an "identity" is false, provide a counterexample.
   a $\alpha (\beta + \alpha) = \alpha \beta + \alpha^*$
   b $(\alpha \beta)^* = \alpha (\beta \alpha)^*$
   c $(\alpha + \beta)^* = (\alpha + \beta)^* \beta^*$
   d $\alpha \alpha = \alpha^*$
   e $(\alpha + \beta)^* = \alpha^* + \beta^*$

5. (10 points) For each of the following Conjectures, either prove it or give a counterexample.
   - **Conjecture A**: For any NFA $N = (Q, \Sigma, \Delta, S, F)$, if $L(N) \neq \emptyset$ then there exists $z \in L(N)$ such that $|z| < |Q|$.
   - **Conjecture B**: For any NFA $N = (Q, \Sigma, \Delta, S, F)$, if $\overline{L(N)} \neq \emptyset$ then there exists $z \in \overline{L(N)}$ such that $|z| < 2^{|Q|}$.

6. (8 points) Is $\{a^i b^j c^k | i \geq 2, j \geq k \geq 0\}$ regular? Justify your answer.
1. For the following algorithm, we interpret $N$ to be a digraph, $G_N$, with vertices $Q$ and an edge $(q,p)$ if $p \in \Delta \{\{q\},x\}$ for some $x \in \Sigma \cup \{\epsilon\}$. We first identify states $q \in Q$ such that there exists $u \in \Sigma^*$ with $q \in \hat{\Delta}(S,u)$. In the following, $\Delta$ must be redefined (restricted) whenever states are removed from $Q$.

   \[
   \text{for each } q \in Q \text{ color } q \text{ red}
   \]

   \[
   \text{for each } q \in S
   \]

   \[
   \text{do a breadth first search in } G_N \text{ and color all reachable states blue}
   \]

   \[
   \text{remove all red states from } Q \text{ and } F
   \]

   We then identify all surviving states $q \in Q$ such that there exists $v \in \Sigma^*$ with $\hat{\Delta}(\{q\},v) \cap F \neq \emptyset$.

   \[
   \text{for each } q \in Q \text{ color } q \text{ red}
   \]

   \[
   \text{for each } q \in Q
   \]

   \[
   \text{for each } p \in F
   \]

   \[
   \text{if there is a path in } G_N \text{ from } q \rightarrow p \text{ then color } q \text{ blue}
   \]

   \[
   \text{remove all red states from } Q \text{ and } F
   \]

2. \(a\) \((1^*10 + 011^*)^* + 1^*
\)

   \(b\) \((1^* + \epsilon)(0 + \epsilon)(1^* + \epsilon)(1^*1011^*)^*(0 + \epsilon)\)

3. The languages are not equivalent. The language of 2 \(a\) contains 1001 and the language of 2 \(b\) does not.

4. \(a\) The identity is false. If $\alpha = 0$ and $\beta = \epsilon$, then $000 \notin L\{\alpha(\beta + \alpha)\} = \{0,0\}$ but $000 \in L\{\alpha\beta + \alpha^*\} = L\{0^*\}$.

   \(b\) The identity is true.

   \(c\) The identity is true.

   \(d\) The identity is false. If $\alpha = 0$, then $00 \notin L\{\alpha\alpha\}$ but $00 \notin L\{\alpha\}$.

   \(e\) The identity is true.
5. **Conjecture A** is true. If $z \in L(N)$, then there exists a path labeled $z$ from some $s \in S$ to some $f \in F$. If $|z| < |Q|$, then we are done. If $|z| \geq |Q|$, then by the Pigeonhole Principle $z$ can be written as $uvw$ such that $|w| \geq 1$ and the state upon consuming $u$ is the same as the state upon consuming $uv$. So there must be a path labeled $uw$ from $s$ to $f$, and $|uw| < |uvw|$.

**Conjecture B** is true. For any NFA $N = (Q, \Sigma, \Delta, S, F)$, we can use the construction from class or the text to derive an equivalent DFA $M$ with $2^{|Q|}$ states. If $L(N) \neq \emptyset$, there must be a state of $M$ which is not a final state. The string $z$ corresponding to the path to the non-final state belongs to $L(N)$, and thus has length $|z| < 2^{|Q|}$.

6. We show that $L = \{a^i b^j c^k | i \geq 2, j \geq k \geq 0\}$ is not regular by invoking the Pumping Lemma. Assume that it is regular. Let $n$ be the constant whose existence is guaranteed by the P.L. Choose $z = a^2 b^n c^n$. By the P.L., $z$ can be expressed as $uvw$ such that $|uv| \leq n$ and $|v| > 0$. There are two cases to consider:

**Case 1:** $z$ contains at least one $a$. By the P.L., $uv^0 w \in L$. But $uv^0 w$ has fewer than 2 $a$'s, so it can't belong to $L$.

**Case 2:** $z$ contains at least one $b$. By the P.L., $uv^0 w \in L$. But $uv^0 w$ has fewer $b$'s than $c$'s, so it can't belong to $L$.

By contradiction, $L$ can't be regular.