1. (8 points) For any NFA \( N = (Q, \Sigma, \Delta, S, F) \), consider \( L_\alpha (N) \) to be the set of all strings \( z \) such that there is a path from every state of \( S \) to every state of \( F \). Prove that \( L_\alpha (N) \) must be regular.

2. (9 points) Give regular expressions for the following languages over \{0, 1\}.
   - a Strings that begin with 01 and end with 10 and contain 11. Note that 0110 belongs to the language.
   - b Strings that contain 01 and contain 10.
   - c Strings in which every 0 is either immediately preceded by a 1 or immediately followed by a 1.

3. (8 points) For each of the following assertions about regular expressions, either prove that it is correct using the identities and rules (9.1) \( \rightarrow \) (9.18) on pp. 49 \( \rightarrow \) 50 of our text or show that it is incorrect.
   - a \( 1^* (011^*)^* + 1^* (011^*)^* 0 = (1+01)^* (\varepsilon + 0) \)
   - b \( 1^* (01^*)^* + 1^* (011^*)^* 0 = (1+01)^* (\varepsilon + 0) \)

4. (14 points) For each of the following languages over \{a, b, c\}, tell whether or not it is regular and justify your response.
   - a \( \{ a^i b^j c^k \mid 0 \leq i \leq j \land 0 < k \} \)
   - b \( \{ a^i b^j c^k \mid 0 \leq i, j \land 0 < k \} \)
1. Given any NFA \( N = (Q, \Sigma, \Delta, S, F) \), construct DFA \( M = \left( 2^\Omega, \Sigma, \delta, S, \{ A \in 2^\Omega \mid A \subseteq F \} \right) \) where for any \( A \in 2^\Omega, a \in A \), \( \delta(A, a) = \bigcup_{q \in A} \Delta(q, a) \).

**Claim:** \( L(N) = L(M) \). **Proof:** (Want to show that \( \forall z \in \Sigma^* \ \hat{\Delta}(S, z) = \hat{\delta}(S, z) \))

\( \varepsilon \in L(N) \Leftrightarrow S \subseteq F \Leftrightarrow \varepsilon \in L(M) \). Assume that \( \hat{\Delta}(S, x) = \hat{\delta}(S, x) \). Then

\( \hat{\Delta}(S, xa) = \Delta(\hat{\Delta}(S, x), a) = \Delta(\hat{\delta}(S, x), a) = \delta(\hat{\delta}(S, x), a) = \hat{\delta}(S, xa) \).

So \( \hat{\Delta}(S, x) \subseteq F \Leftrightarrow \hat{\delta}(S, x) \in \{ A \in 2^\Omega \mid A \subseteq F \} \) and \( L(N) = L(M) \).

2. **a** \( 01(0+1)^*11(0+1)^*10 + 01(0+1)^*10 + 01(0+1)^*110 + 0110 \)

**b** We note that an 01 precedes a 10 (as in the first two terms) or a 10 precedes an 01 (as in the second two terms).

\( (0+1)^*01(0+1)^*10(0+1)^* + (0+1)^*01(0+1)^* + (0+1)^*10(0+1)^*01(0+1)^* + (0+1)^*101(0+1)^* \)

**c** If every 0 is immediately preceded by a 1, then the language is \( (1^*011^*)^* \). If every 0 is immediately followed by a 1, then the language is \( (1^*01)^* \). Or two 0’s can share a 1, thus 010. So our language is \( (1^*(10+010+01))1^*1^* \).

3. **a**

\( 1^*(011^*)^* + 1^*(011^*)^* 0 = 1^*(011^*)^*(\varepsilon + 0) \) \hspace{1cm} (9.7)

\( = (1 + 01)^*(\varepsilon + 0) \) \hspace{1cm} (9.16)

**b** \( 1^*(01)^* + 1^*(011)^* 0 \neq (1 + 01)^*(\varepsilon + 0) \) since 00 belongs to the language \( 1^*(01)^* + 1^*(011)^* 0 \) but not to the language \( (1 + 01)^*(\varepsilon + 0) \).

4. \( \{ a^ib^jc^k \mid 0 \leq i \leq j \land 0 < k \} \) is not regular. If it were, then there would be an \( n \) such that every string \( z \in \{ a^ib^jc^k \mid 0 \leq i \leq j \land 0 < k \} \), \( |z| \geq n \), could be written \( z = uvw \) with \( |uv| \leq n \), \( |v| > 0 \), such that \( \{ uv^i w \mid i \geq 0 \} \subseteq \{ a^ib^jc^k \mid 0 \leq i \leq j \land 0 < k \} \). Choose \( z = a^ib^jc \in \{ a^ib^jc^k \mid 0 \leq i \leq j \land 0 < k \} \). Any attempt to express \( z \) as \( uvw \) with \( |uv| \leq n \) and \( |v| > 0 \) would have \( u \) spanning a nonempty string of \( a \)s and no other letter. So
$uv^2w \in \{a^i b^i c^k \mid 0 \leq i \leq j, 0 < k\}$, which contradicts the conditions of the Pumping Lemma. So $\{a^i b^i c^k \mid 0 \leq i \leq j, 0 < k\}$ can not be a regular language.

$b \{a^i b^i c^k \mid 0 \leq i, j, 0 < k\}$ is regular. It is accepted by the DFA

\[
\left( \{q_a, q_b, q_c, q_{\text{BlackHole}}\}, \{a, b, c\}, \delta, q_a, \{q_c\} \right),
\]

where $q_a$ “means” that the only strings read are $a^*$, $q_b$ “means” that the only strings read are $a^* b b^*$, $q_c$ “means” that the only strings read are $a^* b^* c c^*$, and $q_{\text{BH}}$ is a BlackHole from which one can never escape.

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