1. (10 points) Tell whether each of the following conjectures is true, and justify your answers.

**Conjecture A**: For any DFA \( M = (Q, \Sigma, \delta, S, F) \) there is an NFA \( N = (Q', \Sigma, \Delta, S, F') \) such that \( L(M) = L(N) \) and \( |Q'| \leq |Q| \).

**Conjecture B**: For any NFA \( N = (Q, \Sigma, \Delta, S, F) \) there is a DFA \( M = (Q', \Sigma, \delta, S, F') \) such that \( L(M) = L(N) \) and \( |Q'| \leq |Q| \).

2. (8 points) Show that for any alphabet \( \Sigma \), the set of all strings \( z \) such that there exists \( a \in \Sigma \) and \( a \) does not appear in \( z \) is regular. That is, \( \{ z \mid (\exists a \in \Sigma) z \in (\Sigma - \{a\})^* \} \) is regular. So, if \( \Sigma = \{a, b, c\} \), then \( abab \) and \( \varepsilon \) and \( bbbb \) and \( acaaca \) all belong to the language.

3. (10 points) This problem is a variation of problem 3 of http://web.cs.wpi.edu/~sms/cs3133/HW2-07.pdf

We say that string \( z \) is a *substring of* string \( w \) if there exist strings \( \alpha, \beta \) such that \( w = \alpha z \beta \). Is the following CONJECTURE true? If it is true, then you only need to give a construction to justify your result; you needn't prove that your construction works.

**CONJECTURE**: For any regular language \( L \), then \( Sub(L) = \{ w \mid (\exists z \in L)(w \text{ is a substring of } z) \} \) is regular.
1. **Conjecture A** is true. Given any DFA $M = (Q, \Sigma, \delta, S, F)$, we construct NFA $N = (Q, \Sigma, \Delta, S, F)$ such that for all $q \in Q, a \in \Sigma$, $\Delta(q, a) = \{\delta(q, a)\}$.

**Conjecture B** is false. Let $N = (\{s, q\}, \{0, 1\}, \Delta, \{s\}, \{s, q\})$ where

\[
\begin{align*}
\Delta & = \begin{pmatrix} 0 & 1 \\ s & \{s\} & \{q\} \\ q & \emptyset & \emptyset \end{pmatrix} \\
N \text{ accepts a string if it is a string of 0's followed by at most one 1. Assume there exists a} \text{ two state DFA } M = (\{r, t\}, \{0, 1\}, \delta, \{r\}, F) \text{ to accept } L(N). \text{ Because } \epsilon \in L(N), \text{ it must fol-} \\
& \text{low that } r \in F. \text{ If } \delta(r, 1) = r, \text{ then } 11 \in L(M) \text{ which contradicts the fact that} \\
& 11 \notin L(N). \text{ So } \delta(r, 1) = t. \text{ Because } 1 \in L(N), \ t \in F. \text{ But } 11 \notin L(N), \text{ so } \delta(t, 1) \notin \{r, t\}. \text{ So it fol-} \\
& \text{lows that there does not exist a two state DFA to accept } L(N), \text{ and **Conjecture B** must be false.}
\]

2. We show that the language is regular by constructing an NFA with $|\Sigma| + 1$ states. The NFA first guesses (with $\epsilon$ transitions) which letter $a$ in $\Sigma$ won't appear in the string, and then verifies that $a$ doesn't appear. The acceptor is

\[
(\{s\} \cup \{q_a \mid a \in \Sigma\}, \Sigma, \Delta, \{s\}, \{q_a \mid a \in \Sigma\})
\]

where $\Delta(s, \epsilon) = \{q_a \mid a \in \Sigma\}$ and $(\forall a \in \Sigma) \Delta(s, a) = \emptyset$. For each $a \in \Sigma$, $\Delta(q_a, \epsilon) = \Delta(q_a, a) = \emptyset$ and $\Delta(q_a, u) = \{q_a\}$ for all $u \in \Sigma - \{a\}$.

3. The **Conjecture** is true. If $L = \emptyset$, then $\text{Sub}(L) = \emptyset$, which is regular. Otherwise $L$ is nonempty and there is a DFA $M = (Q, \Sigma, \delta, s, F), \ F \neq \emptyset$, to accept it. We construct a new DFA $M^* = (Q^*, \Sigma, \delta, s, F^*)$ by removing from $Q$ all states which are either not accessible from $s$ or for which there does not exist a path to a state of $F$. We know that $s \in Q^*$, and $L(M) = L(M^*)$. We then construct an NFA $N = (Q^* \cup \{s^*\}, \Sigma, \Delta, \{s^*\}, Q^*)$ with

\[
\begin{align*}
\Delta(s^*, \epsilon) &= Q^*, \ \Delta(s^*, a) = \emptyset \text{ for all } a \in \Sigma, \ \Delta(q, a) = \{\delta(q, a)\} \text{ for all } a \in \Sigma, \ q \in Q^*, \text{ and} \\
\Delta(q, \epsilon) &= \emptyset \text{ for all } q \in Q^*.
\end{align*}
\]