1. (8 points) Show that the language \( \{0^n1^n \mid m \equiv_3 n \} \) is regular by constructing an automaton (DFA or NFA) to accept it, where \( m \equiv_3 n \) means that \( m \) and \( n \) are congruent modulo 3. You don't need to prove that your automaton accepts the language.

2. (8 points) Prove or give a counterexample to the following:

**Conjecture:** For any DFA \( M = (Q, \Sigma, \delta, S, F) \), language \( L(M) \) is infinite if and only if there exists a \( z \in L(M) \) such that \( |Q| \leq |z| < 2^{|Q|} \).

3. (10 points) Prove or give a counterexample to the following:

**Conjecture:** For any regular language \( L \), there exists an NFA \( N = (Q, \Sigma, \Delta, S, F) \) without \( \varepsilon \) transitions to accept \( L \) such that \( |S| = |F| = 1 \).
1. We design an NDA to accept \( \{0^n 10^n : m = n \} \) by first noting that
\[
\{0^n 10^n : m = n \} = \{0^n 10^n : m = n, n \leq 0 \} \cup \{0^n 10^n : m = n, n \leq 1 \} \cup \{0^n 10^n : m = n, n \leq 2 \}.
\]
To test if some \( z \in \{0^n 10^n : m = n \} \) we will use nondeterminism to "guess" to which of the three languages on the right inequality above \( z \) belongs. An acceptor for \( \{0^n 10^n : m = n, n \leq i \}, 0 \leq i \leq 2 \) is \( N_i = \left\{ \{q_{i,0}, q_{i,1}, q_{i,2}, p_{i,0}, p_{i,1}, p_{i,2}\}, \{0,1\}, \Delta, \{q_{i,0}\}, \{p_{i,2}\} \right\} \)
where for \( 0 \leq i, j \leq 2 \),
\begin{itemize}
  \item \( \Delta_i(q_{i,j},0) = \{q_{i,j+1(\text{mod} 3)}\} \),
  \item \( \Delta_i(q_{i,j},1) = \{p_{i,j}\} \),
  \item \( \Delta_i(q_{i,j},1) = \emptyset \) for \( j \neq i \),
  \item \( \Delta_i(p_{i,j},0) = \{p_{i,j+1(\text{mod} 3)}\} \),
  \item \( \Delta_i(p_{i,j},1) = \emptyset \),
  \item \( \Delta_i(q_{i,j},\varepsilon) = \Delta_i(p_{i,j},\varepsilon) = \emptyset \).
\end{itemize}
It is interesting to note that each \( N_i \) is "almost deterministic" (transitions to \( \emptyset \) violate determinism). Finally, we take the union of the three NDAs, and add a new unique start state with an \( \varepsilon \)-transition to each of the start states of \( N_0, N_1 \) and \( N_2 \).
\[
\left\{ \{s\} \cup \{q_{i,j}, p_{i,j} | 0 \leq i, j \leq 2\}, \{0,1\}, \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta^* \{s\}, \{p_{i,j} | 0 \leq i \leq 2\} \right\}
\]
where \( \Delta^*(s,0) = \Delta^*(s,1) = \emptyset \) and \( \Delta^*(s,\varepsilon) = \{q_{i,0} | 0 \leq i \leq 2\} \).

2. The **Conjecture** is true.

Assume that there exists a \( z \in L(M) \) such that \( |Q| \leq |z| \). By the Pigeonhole Principle applied to \( Q \), there must be a decomposition of \( z = uvw \) such that \( |v| \geq 1 \) and 
\[
\hat{\delta}(s,u) = \hat{\delta}(s,uv).
\]
But since \( \hat{\delta}(s,uvw) \in F \), then for all \( k \geq 0 \), \( \hat{\delta}(s,u^kvw) \in F \). So
\[
\{uv^kw | k \geq 0\} \subseteq L(M) \quad \text{and} \quad \{uv^kw | k \geq 0\}
\]
is infinite, so \( L(M) \) must also be infinite.

Assume that \( L(M) \) is infinite. There must be a \( z \in L(M) \) such that \( |z| \geq 2*|Q| \). When considering the states through which \( M \) passes in processing \( z \), some state of \( Q \), say \( q_s \), is traversed at least two times. Consider a shortest path from \( s \) to \( \delta(s,z) \) which passes through \( q_s \), and let \( u \) be the string corresponding to the path from \( s \) to \( q_s \), and let \( v \) be the string corresponding to the path from \( q_s \) to \( \delta(s,z) \). By the minimality of the paths, \( |uv| \leq |Q| \). Consider a shortest nontrivial cycle from \( q \) to \( q \). Let \( x \) be the string corresponding to the shortest nontrivial path from \( q \) to \( q \). We know that for all \( k \geq 0 \),
ux^k \in L(M)$, and there must be a $k \geq 1$ such that $Q \leq |ux^k| < 2*|Q|$, thus establishing the theorem.

3. The Conjecture is false. Assume the Conjecture is true, and choose $L = \{\varepsilon, a\}$. Because $\varepsilon \in L$, the start state must also be the final state. Because $a \in L$, there must be a transition from the start state to the start/final state. But then $aa \in L$, which is a contradiction. So the Conjecture must be false.

   In fact, the Conjecture would be true if $\varepsilon \notin L$. We could introduce a new start state with outgoing edges like those from each of the old start states. We introduce a new final state $f$ and every transition to a final state of the original NFA also includes a transition to $f$, the only final state.