1. (8 points) Let \( L \) be any regular set of binary strings. Prove that the set of strings in \( L \) which contain 00 but do not contain 0101 is regular.

2. (8 points) Prove or give a counterexample to the following:

**Conjecture:** For any NFA \( \mathcal{N} = (Q, \Sigma, \Delta, S, F) \), we obtain an equivalent NFA by contracting any pair of states \( p, q \in Q \) such that \( q \in \Delta(p, \varepsilon) \). That is, if we apply the following transformation to NFA \( \mathcal{N} \) we obtain a new NFA which accepts the same language.

\[
\begin{align*}
\text{if } & \exists p, q \in Q, q \in \varepsilon - \text{closure}(p) \\
& \text{create new state } pq \notin Q \\
& (\forall a \in \Sigma \cup \{\varepsilon\}) \Delta(pq, a) \leftarrow \Delta(p, a) \cup \Delta(q, a) \\
& (\forall r \in Q) (\forall a \in \Sigma \cup \{\varepsilon\}) (p \in \Delta(r, a) \lor q \in \Delta(r, a)) \Rightarrow (\Delta(r, a) \leftarrow \Delta(r, a) \cup \{pq\} - \{p, q\}) \\
& Q \leftarrow Q \cup \{pq\} - \{p, q\} \\
& \text{if } (p \in S) \lor (q \in S) \text{ then } (S \leftarrow S \cup \{pq\} - \{p, q\}) \\
& \text{if } (p \in F) \lor (q \in F) \text{ then } (F \leftarrow F \cup \{pq\} - \{p, q\})
\end{align*}
\]

Intuitively, if there is an \( \varepsilon \)–transition from \( p \) to \( q \), then we replace states \( p \) and \( q \) with a new state, \( pq \), such that all transitions into \( p \) or \( q \) become transitions into \( pq \) and all transitions from \( p \) or \( q \) become transitions from \( pq \), and the new automaton accepts the same language.

3. (10 points) Prove or give a counterexample to the following:

**Conjecture:** For any regular language \( L \), there exists an NFA \( \mathcal{N} = (Q, \Sigma, \Delta, S, F) \) such that:

- \( L = L(\mathcal{N}) \),
- \( |S| = |F| = 1 \),
- \( (\forall q \in Q)(\forall s \in S)(\forall a \in \Sigma \cup \{\varepsilon\})(s \in \Delta(q, a)) \), and
- \( (\forall f \in F)(\forall a \in \Sigma \cup \{\varepsilon\})\Delta(f, a) = \emptyset \).
1. We know that $L_0$, the set of binary strings which contain 00, is regular because it is described by the regular expression $(0+1)^*00(0+1)^*$. We know that $L_1$, the set of binary strings which contain 0101, is regular because it is described by the regular expression $(0+1)^*0101(0+1)^*$. We know that $L_2$, the set of binary strings which do not contain 0101, is regular because regular sets are closed under complement. The strings in $L$ which contain 00 but do not contain 0101 is $L_0 \cap L_2$, and because regular sets are closed under intersection, it must be regular.

2. The CONJECTURE is false. As a counterexample, consider NFA $N = (\{s, p, q\}, \{0,1\}, \Delta, \{s\}, \{p\})$ where

\[
\begin{array}{cccc}
\Delta & 0 & 1 & \epsilon \\
s & \{p\} & \{q\} & \emptyset \\
p & \emptyset & \emptyset & \{q\} \\
q & \emptyset & \emptyset & \emptyset
\end{array}
\]

Applying the transformation yields NFA $N^* = (\{s, pq\}, \{0,1\}, \Delta^*, \{s\}, \{pq\})$ where

\[
\begin{array}{cccc}
\Delta^* & 0 & 1 & \epsilon \\
s & \{pq\} & \{pq\} & \emptyset \\
pq & \emptyset & \emptyset & \{pq\}
\end{array}
\]

But $L(N) = \{0\}$ and $L(N^*) = \{0,1\}$.

3. We establish the CONJECTURE by noting that any regular language has an NFA $N = (Q, \Sigma, \Delta, S, F)$ to accept it. We introduce two new states, $S^*, q_f^* \notin Q$ and we derive $\Delta^*$ from $\Delta$ by adding the transitions $\Delta^*(S^*, \epsilon) = S$, $\Delta^*(S^*, a) = \emptyset$ for all $a \in \Sigma$, $\Delta^*(q_f^*, a) = \emptyset$ for all $a \in \Sigma \cup \{\epsilon\}$, and setting $\Delta^*(q, \epsilon) = \Delta(q, \epsilon) \cup \{q_f^*\}$ for each $q \in F$. So setting $N^* = (Q \cup \{S^*, q_f^*\}, \Sigma, \Delta^*, \{S^*\}, \{q_f^*\})$, we note that $N^*$ satisfies the last three conditions of the CONJECTURE, and that $N$ and $N^*$ have identical transitions within $Q$.

To see that $L(N^*) = L$, we first prove that $L(N^*) \subseteq L$ and then that $L \subseteq L(N^*)$.

If $z \in L(N^*)$, then $q_f^* \in \widehat{\Delta}^*(S^*, z)$. The only transition from $S^*$ is an $\epsilon$-transition to $S$. Since the only transition into $q_f^*$ is $\epsilon$-transition from $F$, the path traced by $\epsilon \epsilon \epsilon = z$ from $S^*$ to $q_f^*$ from a state of $S$ to a state of $F$, implying that $z \in L(N)$. 

If \( z \in L(N) \), there is a path in \( N \) traced by \( z \) from a state \( s \) of \( S \) to a state \( f \) of \( F \). But there is a path in \( N^* \), traced by \( \varepsilon z \varepsilon = z \) from \( S^* \) to \( q_f^* \), which starts with an \( \varepsilon \)–transition from \( S^* \) to \( s \), and finishing with an \( \varepsilon \)–transition from \( f \) to \( q_f^* \). So \( z \in L(N^*) \).