1. (6 points) Produce a DFA which accepts the set of binary strings which do not contain 001 as a substring.

2. (8 points) Concatenation of strings $u,v$ is defined by
   
   - If $|v| = 0$, then $v = \varepsilon$ and $uv = u$.
   - If $|v| > 0$ and $v = wa$, $w \in \Sigma^*$, $a \in \Sigma$, then $uv = (uw)a$.

   Prove that concatenation of strings is associative. That is, for any alphabet $\Sigma$ and for any strings $u, v, w \in \Sigma^*$, $(uv)w = u(vw)$.

3. (8 points) Prove that the following conjecture is false.

   **Conjecture:** For any regular language $L$, there is a DFA $M = (Q, \Sigma, \delta, s, F)$ such that $L = L(M)$ and $|F| = 1$.

   As a hint, find a language $L$ with strings $u, v, w \in \Sigma^*$ such that $u, v, uw \in L$ but $vw \notin L$.

   How would such a machine behave on $u, v, uw$ and $vw$?
1. We first construct a DFA $M_0 = (\{s, a, b, f\}, \{0, 1\}, \delta, s, \{f\})$,

$$
\delta = \begin{array}{c|cc}
& 0 & 1 \\
\hline
s & a & b \\
a & b & s \\
b & b & f \\
f & f & f
\end{array}
$$

which accepts all binary strings which do contain 001 as a substring. The complement of this language is accepted by $M_1 = (\{s, a, b, f\}, \{0, 1\}, \delta, s, \{s, a, b\})$.

2. We prove that $(uv)w = u(vw)$ for all $u, v, w \in \Sigma^*$ by induction on $|w|$.

**Basis** If $|w|=0$, then $w = \varepsilon$, and $(uv)w = (uv)\varepsilon = uv = u(v) = u(v\varepsilon) = u(vw)$

**Induction Hypothesis:** For any $n$ and for any $w, |w| \leq n$, assume that $(uv)w = u(vw)$.

**Induction Step:** For any $x \in \Sigma^*, |x| = n, a \in \Sigma, w = xa$,

$$(uv)w = (uv)xa = ((uv)x)a = u(vx)a = u(v(xa)) = u(vw)$$

where the 3rd equality follows from the induction hypothesis.

3. To prove that the conjecture is false, consider $L = \{0, 1, 00\}$ over $\Sigma = \{0, 1\}$, and assume that there is a DFA $M = (Q, \Sigma, \delta, s, F)$ such that $L = L(M)$ and $|F| = 1$. Without loss of generality, let $F = \{f\}$. Since $\{0, 1\} \subseteq L$, it must follow that $\delta(s, 0) = \delta(s, 1) = f$. And since $00 \in L$, then $\hat{\delta}(s, 00) = \delta(f, 0) = f$. But since $10 \notin L$, then $\hat{\delta}(s, 10) = \delta(f, 0) \neq f$. But this, $\delta(f, 0) \neq \delta(f, 0)$ is a contradiction, so such a machine cannot exist.