CS584

MIDTERM EXAM

Name______________

Date: October 20, 2008
All documentation permitted

1. (15 points) Consider the problem of finding the MAXIMUM **and** the MINIMUM of *n* numbers within the computational model of using pairwise comparisons as our benchmark operation. What lower bound is obtained for the worst case time to solve the problem by using the decision tree argument? Explain your bound. Give an exact number of comparisons (do not use asymptotic notation).

2. (20 points) Assume you are given an unsorted list A of $n \geq 2$ of integers. Use pairwise comparisons as the benchmark operation.

a How can you find, in worst-case $O(n)$ time, $x, y \in A$ such that $|x-y| \geq |w-z|$ for all $w, z \in A$?

b How can you find, in worst-case $O(n \lg n)$ time, $x, y \in A$, $x \neq y$, such that $|x-y| \leq |w-z|$ for all $w, z \in A$?

3. (30 points) Suppose that you want to find the index of the maximum element of list $A[1..n]$ using the program

```
MAX ← 1
for k ← 2 to n
    if A[k]>A[MAX] then MAX ← k
```

Assuming that every element of A is distinct, what is the expected number of times that $\text{MAX} \leftarrow k$ is executed according to each of the following distributions.

a Every permutation of A is equally likely.

b $\Pr\{A[i] < A[k], 1 \le i < k\} = \dfrac{1}{k * H_n}$ for $2 \le k \le n$.

4. (20 points) Prove or give a counterexample to the following CONJECTUREd Corollaries to the Blue and Red Rules.

CONJECTURE *a*: Let $G=(V,E)$, $w:E\to\Re^{+}$, be a weighted connected graph. Let *C* be any cut of *G* with at least two edges, and let *e* be the heaviest edge of the cut. That is, $w(e)=\max_{e'\in C}\{w(e')\}$. Increasing the weight of *e* will not increase the weight of the MST of *G*.

CONJECTURE *b*: Let $G=(V,E)$, $w:E\to\Re^{+}$, be a weighted connected graph. Let *C* be any cycle of *G*, and let *e* be the heaviest edge of the cycle. That is, $w(e)=\max_{e'\in C}\{w(e')\}$. Increasing the weight of *e* will not increase the weight of the MST of *G*.

5. (15 points) For DIJKSTRA'S ALGORITHM we assume that all edge weights in a digraph are positive, $w: E \rightarrow \Re^{+} \ \ \forall e \in E$. Can we relax this assumption? That is, suppose that we allow $w: E \rightarrow \Re \ \forall e \in E$. Does DIJKSTRA'S ALGORITHM still work?

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Solutions to Midterm Exam

1. Any decision tree to solve this problem must have at least one leaf corresponding to every pair of distinct input numbers (since any pair of numbers can correspond to MAXIMUM and MINIMUM). Hence, any decision tree must have at least $n(n-1)$ leaves. The worst case number of comparisons corresponds to a deepest leaf, and the depth of the tree must be at least the logarithm (to the base 2) of the number of leaves. Hence, the worst case number of comparisons must be at least $\lceil \lg(n(n-1)) \rceil = \lceil \lg n + \lg(n-1) \rceil \approx 2\lg n$.

2. ***a***

```
x ← MAX(A)                                              O(n)
y ← MIN(A)                                              O(n)
```

b

```
HEAPSORT(A)                                             O(nlgn)
                  /* We know that x and y must now be adjacent in A */
closest ← 0       /* A[closest] & A[closest+1] is closest pair so far */
for i ← 1 to n-1 do
    if ( (A[i+1]-A[i]) < (A[closest+1] - A[closest]) )
    then closest ← i                                    O(n)
return A[closest], A[closest+1]
```

3. Letting rv $X_k = \begin{cases} 1, \text{if } A[i] < A[k] \text{ for } 2 \le i < k \\ 0, \text{otherwise} \end{cases}$ for $2 \le k \le n$, we find that the number of executions of $Max \leftarrow k$ is $\sum_{2\le k\le n} X_k$. The problem asks for

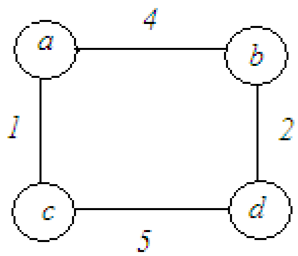
$$E\left[\sum_{2\le k\le n} X_k\right] = \sum_{2\le k\le n} E[X_k] = \sum_{2\le k\le n} \Pr\{X_k = 1\}.$$

a $\Pr\{X_k = 1\} = 1/k$, so $E\left[\sum_{2\le k\le n} X_k\right] = \sum_{2\le k\le n} 1/k = \sum_{1\le k\le n} 1/k - 1 = H_n - 1$

b $\Pr\{X_k = 1\} = \dfrac{1}{k * H_n}$, so

$$E\left[\sum_{2\le k\le n} X_k\right] = \sum_{2\le k\le n} \frac{1}{k * H_n} = \frac{1}{H_n}\sum_{2\le k\le n}\frac{1}{k} = \frac{1}{H_n}\left(\sum_{1\le k\le n}\frac{1}{k} - 1\right) = \frac{H_n - 1}{H_n} = 1 - \frac{1}{H_n}$$

4. CONJECTURE ***a*** is false. In graph



Edge *bd* is the heaviest edge in cut $\{\{a,b\},\{c,d\}\}$, but increasing its weight from 2 to 3 increases the weight of the MST from 7 to 8.

CONJECTURE ***b*** is true. For any edge *e* removed (colored red) by the Red Rule, we know that *G* admits an MST which doesn't contain *e*. So increasing $w(e)$ does not increase the weight of the MST.

5. The algorithm doesn't work. For the following graph, when it EXTRACTs *a* from the priority queue, it has $d[a]=10$. However the length of a shortest path from σ to *a* is 5.