CS584

FINAL EXAM

Date: December 16, 2009 Name____________________

All documentation permitted

1. (20 points) Professor K has a CONJECTURE. Tell whether or not he is correct, and justify your answer.

> PROFESSOR K'S **CONJECTURE**: Given a directed weighted graph with source $\sigma \in V$ in which every edge has a positive weight except possibly for edges from σ, DIJKSTRA'S ALGORITHM will **always** find the shortest path from σ to every vertex. Just to clarify, all edge weights are positive except possibly edges of $E \cap (\{\sigma\} \times V)$.

2. (30 points) Assume you are given text file, $Text[1..n]$, from which all punctuation and spacing have been removed. For example, it could contain *tobeornottobethatisthequestion*. All legal words are stored in an English dictionary which supports queries of the form $Dict?[i, j], 1 \le i \le j \le n$ in time in $O(1)$. For example, *Dict?*[3,4] would return true because *be* is a word, and *Dict?*[3,8] would return false because *beorno* is not a word.

a Describe an algorithm with time complexity in $O(n^2)$ to test if *Text* can be decomposed into a sequence of English words.

b Describe an algorithm with time complexity in $O(n^2)$ to find the minimal number of English words into which *Text* can be decomposed.

3. (15 points) In randomized table lookup, we use a sequence of functions $f_0, f_1, f_2, \ldots$ to distribute keys independently uniformly over a sequence of bins, $b_0, b_1, \ldots, b_{p-1}$, where each bin can contain at most one key. Assuming there are k keys already in the bins, what is the expected number of probes of bins before finding one that does not contain a key?

4. (15 points) Suppose you are given a weighted graph $G=(V,E)$, $\sigma,\tau\in V$, with positive weights on the edges **and** the vertices. The *length* of a path is the sum of the weights of the vertices plus the sum of the weights of the edges of the path. Describe an algorithm with execution time in $O\left(|V|^2\right)$ to find a shortest path from σ to τ.

5. (20 points) A *spanning path* of a graph is a path that goes through every vertex of the graph. A *Hamiltonian cycle* of a graph is a cycle that goes through every vertex of the graph. Show that the problem of deciding if a graph has a spanning path is polynomially reducible to the problem of deciding if a graph has a Hamiltonian cycle.

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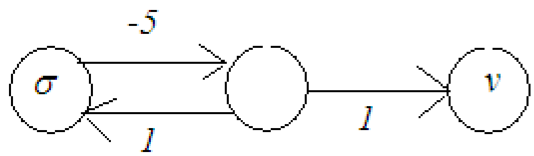
Solutions to Final Exam

1. Either of the following answers is acceptable, depending upon whether paths must be simple (never visit the same vertex twice).

· The CONJECTURE is correct. When a vertex is labeled (removed from the priority queue), its *D* value is correct (it's the length of a shortest path from σ to the vertex). It can't be shortened because any edge (aside from edges of the form σv) have positive weights.

· The CONJECTURE is not correct because the following graph does not have a shortest path from σ to *v*.



2. We use array *MinNum*[0..*n*] to contain 0 in case *Text*[1..*k*] can't be decomposed into a sequence of words, and otherwise it contains the minimum number of words.

$MinNum[0] \leftarrow 1$
for $k \leftarrow 1$ **to** n **do** /* compute *MinNum*[*k*] */
 $MinNum[k] \leftarrow \infty$
 for $j \leftarrow 0$ **to** k-1 **do**
  if $(MinNum[j] > 0) \wedge Dict?[j+1, k]$ /* *MinNum*[*k*]>0 */
  then $MinNum[k] \leftarrow \min(MinNum[k], 1 + MinNum[j])$
 if $MinNum[k] = \infty$ **then** $MinNum[k] \leftarrow 0$
return $MinNum[n]$

3. Letting rv *X* denote the number of probes of bins, we note that since the probes are independent with probability of success (finding an empty bin) equal to $\frac{p-k}{p}$, then *X* is geometrically distributed. And $E[X] = \frac{p}{p-k}$.

4. Replace each vertex $v \in V$ with a pair of vertices, v_{in} and v_{out}, and an edge between v_{in} and v_{out} of weight *w*(*v*). There are no costs associated with vertices in the new graph. Edges incident to *v* are now incident to v_{in} and v_{out}. The shortest path from σ to τ in the original graph corresponds to the shortest path from σ_{in} to τ_{out} in the new graph, and it

has the same cost. We find this path using Dijkstra's Algorithm, which has a complexity in $O\left(|V|^2\right)$.

5. Take any instance $G=(V,E)$ of the spanning path problem. Construct a new graph $G^*=\left(V\cup\{w\},E^*\right)$ where E^* is E plus an edge between w and every $v\in V$. G has a spanning path if and only if G^* has a Hamilton cycle, and the spanning path of G is obtained from the Hamilton cycle of G^* by removing both edges incident with w.