CS584

FINAL EXAM

Date: December 10, 2008 Name____________________

All documentation permitted

1. (10 points) A bureaucracy has *m* people and needs *n* committees. For each person $p_i, 1 \leq i \leq m$, we have a list of committees on which p_i is capable of serving and a number $x_i \in \mathbb{Z}^{\geq 0}$ of committees on which p_i is willing to serve. For each committee $c_j, 1 \leq j \leq n$, we have a number $y_j \in \mathbb{Z}^{+}$ of members needed on c_j. Describe a polynomial time algorithm to determine if an assignment of people to committees is possible under the given constraints, and to find a feasible assignment if one exists.

2. (20 points) Given array $A[1..n]$ of distinct integers, describe an algorithm of complexity $O\left(n^2\right)$ to find the length of a longest (not necessarily contiguous) increasing sequence of integers of A. For example, if A=(11, 17, 5, 8, 6, 4, 7, 12, 3), then the answer would be 4 because of the subsequence 5, 6, 7, 12 .

3. (5 points) Is the following conjecture true or false? If it is false, give a counterexample. That is, give a network, a flow, and two cuts with unequal flows.

<u>CONJECTURE</u>: For any network and any flow, the flows through all cuts are equal.

4. (20 points) Suppose you are given a network $G=(V,E)$, $\sigma,\tau \in V$, $c:E\rightarrow\mathbb{Z}^+$, and max flow $f:E\rightarrow\mathbb{Z}^+$. You are also given an $uv \in E$. Find a max flow in the network $G^*=(V,E)$, $\sigma,\tau \in V$, $c^*:E\rightarrow\mathbb{Z}^+$ such that $c^*(e)=\begin{cases} c(e), \text{if } e \neq uv \\ c(e)-1, \text{if } e = uv \end{cases}$. Describe an algorithm to find a max flow in G^*. Your algorithm should work in worst-case time in $O(m+n)$.

5. (25 points) Given digraph G represented by $n \times n$ adjacency array A, where

$$A[i, j] = \begin{cases} 1, \text{ if edge } ij \in E \\ 0, \text{ if edge } ij \notin E \end{cases}$$

and integer $k \geq 2$, design an algorithm to compute $n \times n$ array B such that $B[i, j]$ contains the number of paths of **even length** from i to j such that the length is less than or equal to k. Your algorithm should work in time in $O(kn^3)$.

6. (20 points) (Probabilistic Counting) Consider the following algorithm to estimate the size of a set S. Assume that you know integers $0<k,l\leq|S|$.

> Sample (without replacement from a uniform distribution) and mark k elements of S
> Replace the k elements
> Sample (with replacement from a uniform distribution) l elements of S
> Let m be the number of marked elements from the second sample

Describe an unbiased estimator for $|S|$, and show that your estimator is unbiased.

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Solutions to Final Exam

1. We form a flow network with vertices $\{p_1,...,p_m,c_1,...,c_n,\sigma,\tau\}$ and edges $\{\sigma p_i \mid 1 \le i \le m\} \cup \{c_j\tau \mid 1 \le i \le n\} \cup \{p_i c_j \mid p_i \text{ can serve on } c_j\}$. The capacity of edge σp_i is x_i and the capacity of edge $c_j\tau$ is y_j. The capacities of all other edges is 1. A max flow f is a feasible assignment if $|f| = \sum_{1\le j\le n} y_j \le \sum_{1\le i\le m} x_i$.

2. $Length\_of\_Longest[1] \leftarrow 1$

  for $i \leftarrow 2$ **to** n **do**

    $Length\_of\_Longest[i] \leftarrow 1 + \max_{\substack{1\le j<i \\ A[j]<A[i]}} \left(Length\_of\_Longest[j]\right)$

  return $\max_{1\le i\le n}\left(Length\_of\_longest[i]\right)$

3. The CONJECTURE is true.

4. We first determine a feasible flow, f^*, which is feasible in G^*, and then we try to find an augmenting path relative to f^*.

if $f(uv) < c(uv)$ **then return** f

Use dfs to find a path π_0 with positive flow on each edge from σ to u

Use dfs to find a path π_1 with positive flow on each edge from v to τ

for each edge $e \in \pi_0 uv \pi_1$ **do** $f(e) \leftarrow f(e) - 1$

Construct $H = (V, E')$ with capacities c^* such that $\forall e \in E'$ $c^*(e) = c(e) - f(e)$

Use dfs to find a path π (if it exists) from σ to τ consisting only of edges with positive values of c^*

return $f^*(e) = \begin{cases} f(e)+1, \text{ if } e \in \pi \\ f(e), \text{ otherwise} \end{cases}$

5. $A^2 \leftarrow AA$

  $B \leftarrow A^2$

  for $l \leftarrow 4$ **to** k **by** 2 **do**

    $B \leftarrow B + BA^2$

  return B

6. Since this sampling of the *l* elements selected in the second sample is with replacement, the probability of each sampled element being marked is $k/|S|$. The number of marked elements is binomially distributed, with the expected number of marked elements being $\sum_{1\le j\le l}\binom{l}{j}\left(\frac{k}{|S|}\right)^{j}\left(1-\frac{k}{|S|}\right)^{l-j}=l\frac{k}{|S|}=E[m]$. So we infer $|S|=\frac{lk}{m}$.