1. (10 points) A bureaucracy has \( m \) people and needs \( n \) committees. For each person \( p_i, 1 \leq i \leq m \), we have a list of committees on which \( p_i \) is capable of serving and a number \( x_i \in \mathbb{Z}^{\geq 0} \) of committees on which \( p_i \) is willing to serve. For each committee \( c_j, 1 \leq j \leq n \), we have a number \( y_j \in \mathbb{Z}^+ \) of members needed on \( c_j \). Describe a polynomial time algorithm to determine if an assignment of people to committees is possible under the given constraints, and to find a feasible assignment if one exists.
2. (20 points) Given array $A[1..n]$ of distinct integers, describe an algorithm of complexity $O(n^2)$ to find the length of a longest (not necessarily contiguous) increasing sequence of integers of $A$. For example, if $A=(11, 17, 5, 8, 6, 4, 7, 12, 3)$, then the answer would be 4 because of the subsequence 5, 6, 7, 12.
3. (5 points) Is the following conjecture true or false? If it is false, give a counterexample. That is, give a network, a flow, and two cuts with unequal flows.

**Conjecture:** For any network and any flow, the flows through all cuts are equal.

4. (20 points) Suppose you are given a network $G=(V,E)$, $\sigma, \tau \in V$, $c: E \to \mathbb{Z}^+$, and max flow $f: E \to \mathbb{Z}^+$. You are also given an $uv \in E$. Find a max flow in the network $G^*=(V,E)$, $\sigma, \tau \in V$, $c^*: E \to \mathbb{Z}^+$ such that $c^*(e) = \begin{cases} c(e), & \text{if } e \neq uv \\ c(e) - 1, & \text{if } e = uv \end{cases}$. Describe an algorithm to find a max flow in $G^*$. Your algorithm should work in worst-case time in $O(m+n)$. 


5. (25 points) Given digraph $G$ represented by $n \times n$ adjacency array $A$, where

$$A[i,j] = \begin{cases} 1, & \text{if edge } ej \in E \\ 0, & \text{if edge } ej \notin E \end{cases}$$

and integer $k \geq 2$, design an algorithm to compute $n \times n$ array $B$ such that $B[i,j]$ contains the number of paths of even length from $i$ to $j$ such that the length is less than or equal to $k$. Your algorithm should work in time in $O(kn^3)$. 
6. (20 points) (Probabilistic Counting) Consider the following algorithm to estimate the size of a set $S$. Assume that you know integers $0 < k, l \leq |S|$. 

Sample (without replacement from a uniform distribution) and mark $k$ elements of $S$
Replace the $k$ elements
Sample (with replacement from a uniform distribution) $l$ elements of $S$
Let $m$ be the number of marked elements from the second sample

Describe an unbiased estimator for $|S|$, and show that your estimator is unbiased.
1. We form a flow network with vertices \( \{ p_1, \ldots, p_m, c_1, \ldots, c_n, \sigma, \tau \} \) and edges 
\( \{ \sigma p_i \mid 1 \leq i \leq m \} \cup \{ c_j \tau \mid 1 \leq j \leq n \} \cup \{ p, c_j \mid p_i \text{ can serve on } c_j \} \). The capacity of edge \( \sigma p_i \) is \( x_i \) and the capacity of edge \( c_j \tau \) is \( y_j \). The capacities of all other edges is 1. A max flow \( f \) is a feasible assignment if \( |f| = \sum_{i,j \in \mathcal{M}} y_j \leq \sum_{i \in \mathcal{M}} x_i \).

2. \( \text{Length of Longest}[1] \leftarrow 1 \)
   for \( i \leftarrow 2 \) to \( n \) do
   \( \text{Length of Longest}[i] \leftarrow 1 + \max \{ \text{Length of Longest}[j] \mid \frac{i}{j} < \frac{i}{j} \leq \frac{i}{j} \} \)
   return \( \max_{i \in \mathcal{I}} \{ \text{Length of longest}[i] \} \)

3. The CONJECTURE is true.

4. We first determine a feasible flow, \( f^* \), which is feasible in \( G^* \), and then we try to find an augmenting path relative to \( f^* \).

   if \( f(uv) < c(uv) \) then return \( f \)

   Use dfs to find a path \( \pi_0 \) with positive flow on each edge from \( \sigma \) to \( u \)

   Use dfs to find a path \( \pi_1 \) with positive flow on each edge from \( v \) to \( \tau \)

   for each edge \( e \in \pi_0 \cup \pi_1 \) do \( f(e) \leftarrow f(e) - 1 \)

   Construct \( H = (V, E') \) with capacities \( c^* \) such that \( \forall e \in E' \ c^*(e) = c(e) - f(e) \)

   Use dfs to find a path \( \pi \) (if it exists) from \( \sigma \) to \( \tau \) consisting only of edges with positive values of \( c^* \)

   return \( f^*(e) = \begin{cases} f(e) + 1, & \text{if } e \in \pi \\ f(e), & \text{otherwise} \end{cases} \)

5. \( A^2 \leftarrow AA \)
   \( B \leftarrow A^2 \)
   for \( l \leftarrow 4 \) to \( k \) by \( 2 \) do
   \( B \leftarrow B + BA^2 \)
   return \( B \)
6. Since this sampling of the $l$ elements selected in the second sample is with replacement, the probability of each sampled element being marked is $k / |S|$. The number of marked elements is binomially distributed, with the expected number of marked elements being

$$\sum_{l \leq j \leq l} \binom{l}{j} \left( \frac{k}{|S|} \right)^j \left( 1 - \frac{k}{|S|} \right)^{l-j} = \frac{k}{|S|} = E[m].$$

So we infer $|S| = \frac{lk}{m}$.