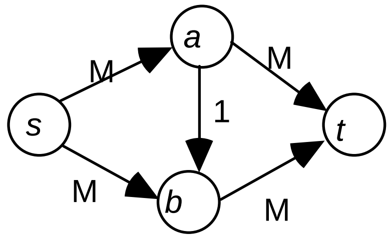
Analysis

Even if capacities are integral, may be real slow: initially, $f(e)=0$ for all $e \in E$



Augmenting paths: *s-a-b-t, s-b-a-t, s-a-b-t,...* +1 every time
need $2M$ augmentations before flow of M reached.

There is a case with irrational capacities for which the algorithm neither terminates nor converges.

ANALYSIS: If $c: E \to \mathbb{Z}^+$, then FF halts in at most max-flow augmentations, or time $O(|E|\kappa)$, where $\kappa := \max_{e \in E} c(e)$.

Edmonds & Karp: Choose augmenting path with
- Max bottleneck capacity
- sufficiently large capacity (CAPACITY SCALING)
- SHORTEST # OF HOPS

CAPACITY SCALING:

$G(\Delta)$:= G with all edges of residual capacity or flow $\geq \Delta$

for each $e \in E$ $f(e) \leftarrow 0$

$\Delta \leftarrow \max_m \left(\exists i \left(m = 2^i\right) \wedge \left(m \leq \kappa\right)\right)$

while $\Delta \geq 1$

  while $\exists$ augmenting path P in $G(\Delta)$

    $f \leftarrow$ AUGMENT(f,P)

  $\Delta \leftarrow \Delta / 2$

return f

ANALYSIS:
- returned f is maximum.
- Outer **while** repeats $\lfloor \lg \kappa \rfloor + 1$ times.
- With flow f after a Δ-phase, max flow $\leq |f| + \Delta |E|$
- Each augmentation in Δ-phase increases flow by $\geq \Delta$
- Each Δ-phase admits $\leq |E|$ augmentations.
- Each augmentation takes time in $O(|E|)$.

Thus CAPACITY SCALING runs in time $O\left(|E|^2 \lg \kappa\right)$.

SHORTEST # OF HOPS

The *level graph of G*, G_l, is breadth first search graph of G with sideways and back edges deleted. The *level* of vertex v is length of shortest path $s \to v$.

Any shortest path $s \to v$ in G is a path of G_l.

CLAIM: • Let P be shortest augmenting path of G, G' the residual graph after augmenting along P, and Q be shortest augmenting path of G', then $|Q| \geq |P|$.

• Augmenting along shortest paths, after $\leq |E|$ augmentations the length of a shortest path must increase strictly.

PROOF (of CLAIM): Any path using back or side edge of G_l strictly longer than P. After any augmentation, ≥ 1 edge is saturated (or gets 0 flow on backedge) & moves deeper into G_l. This can occur $\leq |E|$ times.

ANALYSIS: $\exists \leq |E| * |V|$ augmentations, & using BFS each augmentation $O(|E|)$, so $O\left(|E|^2 |V|\right)$.

THEOREM: (Edmonds & Karp) If augmenting paths sought by breadth-first search (augmenting path with minimal number of arcs chosen), then Ford-Fulkerson halts after $\leq \frac{|V| * |E|}{2}$ augmentations.

State of the Art:

Dantzig '51	Simplex LP	$O\left(|E||V|^2 \kappa\right)$
Ford Fulkerson '55	Augmenting Path	$O(|E||V|\kappa)$
Edmonds Karp '70	Shortest Path	$O\left(|E|^2 |V|\right)$
Dinitz '70	Shortest Path	$O\left(|E||V|^2\right)$
Edmonds Karp Dinitiz '72	Capacity Scaling	$O\left(|E|^2 \lg \kappa\right)$
Dinitz Gabow '73	Capacity Scaling	$O(|E||V| \lg \kappa)$
Karzanov '74	Preflow Push	$O\left(|V|^3\right)$
Sleator Tarjan '83	Dynamic trees	$O(|E||V| \lg |V|)$
Goldberg Tarjan '86	FIFO Preflow Push	$O\left(|E||V| \lg\left(\frac{|V|^2}{|E|}\right)\right)$
Goldberg Rao '97	Length function	$O\left(|E|^{3/2} \lg\left(\frac{|V|^2}{|E|}\right) \lg \kappa\right)$ $O\left(\frac{|E|}{|V|^{2/3}} \lg\left(\frac{|V|^2}{|E|}\right) \lg \kappa\right)$

EX: *Maximum Matching in Bipartite Graphs*

If (undirected) graph $G = (V, E)$ satisfies $V = X \cup Y, X \cap Y = \emptyset, E \subseteq X \times Y$, a *matching* is $M \subseteq E$ satisfies each $v \in V$ belongs to at most one $e \in M$.

ALGORITHM: Construct network $\left(V \cup \{s, t\}, E \cup (\{s\} \times X) \cup (\{t\} \times Y)\right)$ with capacities $(\forall e \in E) c(e) = 1$ and $(\forall e \in \{s\} \times X \cup \{t\} \times Y) c(e) = 1$

Ford-Fulkerson returns max flow with all integral flows. $(\forall e \in E) e \in M \Leftrightarrow f(e) = 1$

EX: Menger's Theorem – The maximum number of edge disjoint paths joining 2 vertices in a digraph = the minimaum number of edges whose removal separates the vertices.
PROOF: Label the vertices *s* & *t*, and assign capacity 1 to every edge. $\exists k$ edge disjoint paths iff $\exists$ flow *f* with $|f| = k$. MaxFlow-MinCut $\Rightarrow \exists$ cut of capacity *k* iff *k* max flow.

EX: A *network* is:

-a digraph (V,E),
-*capacity* $c:E \to \mathbf{R}^+$,
-*demands* $d:V \to \mathbf{R}$

A *circulation*, $f:E \to \mathbf{R}^+$ satisfies:

CAPACITY CONSTRAINT- $(\forall e \in E) 0 \le f(e) \le c(e)$
FLOW CONSERVATION- $\forall v \in V \sum_u f(u,v) - \sum_u f(v,u) = d(v)$

QUESTION: Does there exist a circulation?
Necessary Condition for existence: $\sum_{\substack{v\in V \\ d(v)>0}} d(v) = \sum_{\substack{v\in V \\ d(v)<0}} -d(v)$

Solution:
- Add to *V* two vertices $s,t \notin V$
- $\forall v \in V \ d(v) < 0$ add to *E* edge *sv* with capacity $-d(v)$
- $\forall v \in V \ d(v) > 0$ add to *E* edge *vt* with capacity $d(v)$

$\exists$ circulation iff the extended network has max flow $|f| = \sum_{\substack{v\in V \\ d(v)>0}} d(v)$

EX: Circulation with demands and lower bounds. A *network* is:

-a digraph (V,E),
-*capacity* $c:E \to \mathbf{R}^+$,
-*lower bounds* $\lambda:E \to \mathbf{R}^+$,
-*demands* $d:E \to \mathbf{R}$

A *circulation*, $f:E \to \mathbf{R}^+$ satisfies:

CAPACITY CONSTRAINT- $(\forall e \in E) \lambda(e) \le f(e) \le c(e)$
FLOW CONSERVATION- $\forall v \in V \sum_u f(u,v) - \sum_u f(v,u) = d(v)$

QUESTION: Does there exist a circulation satisfying lower bounds?
(Idea: Model lower bound $\lambda(uv)$ as demand -$\lambda(uv)$ from *v* and $\lambda(uv)$ from *u*.)
Create network *G*' with capacities $c'(e) = c(e) - \lambda(e), \forall e \in E$ and demands
$d'(u) = d(u) + \sum_{uv\in E} \lambda(uv) - \sum_{vu \in E} \lambda(vu)$.

THEOREM: $\exists$ circulation in *G* $\Leftrightarrow \exists$ circulation in *G*'.
"PROOF": *f* is circulation in *G* $\Leftrightarrow f - \lambda$ is circulation in *G*'.