CS584
HW#6

DUE: Monday, December 14

1. (14 points) Assume that in the STABLE MARRIAGE PROBLEM each of *n* men ranks (in a total order) all of *n* women drawing independently from a uniform distribution over the set of all permutations of the women, and each of *n* women ranks (in a total order) all of *n* men drawing independently from a uniform distribution over the set of all permutations of the men. For some fixed *k*, $1 \le k \le n$, let us define the *k*-COOL PROPERTY:

> There is a set of *k* men, M_{cool}, and *k* women, W_{cool}, such every man's first *k* women all belong to W_{cool}, and every woman's first *k* men all belong to M_{cool}.

a What is the probability that the rankings satisfy the *k*-COOL PROPERTY?

b Assuming that the rankings are drawn as specified above and that they satisfy the *k*-COOL PROPERTY and that the men propose according to the MALE PROPOSAL STRATEGY, give an upper bound on the expected depth that each $m \in M_{cool}$ goes into his list. Justify your response.

2. (4 points) Prove or give a counterexample to the following.

CONJECTURE: If (*S*,*T*) is the unique minimum cut in a flow network with integral capacities, then (*S*,*T*) will still be the minimum cut even if every capacity is increased by 1.

3. (8 points) Our old friend Professor K is still algorithmically clueless, and cheap. He still has *n* sensors distributed in space, but they are mobile. And the *k* transmitters are now fixed in space. Each sensor can communicate with any transmitter which is close enough, where distance is Euclidean distance. But any transmitter can send to the base station information from up to λ sensors. Design an efficient algorithm to find if there is a feasible assignment of sensors to transmitters for instances of the following problem:

INPUT: Locations of *n* sensors, $\left(x_{s_1}, y_{s_1}\right), \ldots, \left(x_{s_n}, y_{s_n}\right)$

Locations of *k* transmitters, $\left(x_{t_1}, y_{t_1}\right), \ldots, \left(x_{t_k}, y_{t_k}\right)$

Distance threshold, θ

Maximum transmitter load, λ

OUTPUT: Assignment of sensors to transmitters, $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$

such that $\sqrt{\left(x_{s_i} - x_{f(s_i)}\right)^2 + \left(y_{s_i} - y_{f(s_i)}\right)^2} \le \theta$ for $1 \le i \le n$ and

$\left|\{i \mid f(i) = j\}\right| \le \lambda$ for $1 \le j \le k$.

4. (8 points) The problem of deciding if you can divide n integers $\{p_1, ..., p_n\}$ into two sets with equal sums is NP-complete. Show that the KNAPSACK PROBLEM is NP-complete.

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SOLUTIONS

1. ***a*** Fixing k, there are $\binom{n}{k}$ ways to choose M_{cool}, and $\binom{n}{k}$ ways to choose W_{cool}. For each choice of M_{cool} and W_{cool}, for any man m (respecively w), $k!(n-k)!$ of m's $n!$ possible rankings of the women put all the women in W_{cool} first, and likewise for each woman. So the probability that m's rankings start with every woman in W_{cool} is

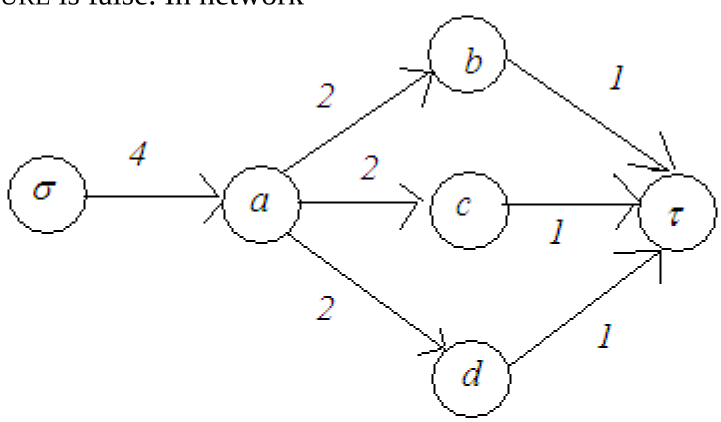
$$\frac{k!(n-k)!}{n!} = \frac{1}{\binom{n}{k}}.$$

And because the rankings are chosen independently, the probability that all $2n$ men and women satisfy the *cool*-condition for the particular choice of M_{cool} and W_{cool} is $\binom{n}{k}^{2n}$.

Since there are $\binom{n}{k}^2$ possible choices of M_{cool} and W_{cool}, the answer is $\binom{n}{k}^{2-2n}$.

b The subproblem defined on M_{cool} and W_{cool} comprises its own STABLE MARRIAGE PROBLEM, and as we showed in class letting amnesiac men propose, the average cool man goes $\leq H_k$ into his list.

2. The CONJECTURE is false. In network



the minimum cut is $(\{\sigma, a, b, c, d\}, \{\tau\})$. But after adding 1 to every capacity, the new minimum cut is $(\{\sigma\}, \{a, b, c, d, \tau\})$.

3. We translate an instance of the problem to a max-flow problem. The network in which we seek a flow of value n has vertices $\{\sigma, s_1, ..., s_n, t_1, ..., t_k, \tau\}$, There are edges $\{\sigma\} \times \{s_1, ..., s_n\}$, all of capacity 1, edges (s_i, t_j) for all i and j such that

$\sqrt{\left(x_{s_i} - x_{t_j}\right)^2 + \left(y_{s_i} - y_{t_j}\right)^2} \leq \theta$ all of capacity 1, and edges $\{t_1, ..., t_k\} \times \{\tau\}$ all of capacity λ.

4. We convert any instance $\{p_1, ..., p_n\}$ of our problem (the PARTITION PROBLEM) into an instance of the KNAPSACK PROBLEM in which object *i*, $1 \leq i \leq n$, has value **and** weight p_i. The capacity of the knapsack is $\frac{\sum_{1 \leq i \leq n} p_i}{2}$, and the instance of PARTITION PROBLEM admits a solution if and only if the KNAPSACK PROBLEM admits a solution of value $\frac{\sum_{1 \leq i \leq n} p_i}{2}$.