CS584
HW#6

Due: December 8

1. (2 points) Do **Exercise *26.1-6*** of page 650 of CLRS, where the properties are the **Capacity Constraint**, **Skew Symmetry** and **Flow Conservation**.

2. (6 points) Given a directed graph $G=(V,E)$, a positive integer k and a pair of vertices $x,y\in V$, we want to decide if there exist k edge disjoint paths from x to y. Show how to solve this problem in polynomial time.

3. (6 points) Do **Exercise *26.2-9*** of page 664 of CLRS. In the statement of the problem, replace "...at most $|V|$ flow networks..." with "...at most $|V|^2$ flow networks...".

4. (5 points) Prove or give a counterexample to the following.
   CONJECTURE: Assume that for arbitrary network $G=(V,E), \sigma,\tau\in V$, with capacities $c:E\to\Re^+$, cut (S,T) is a minimum cut. In network $G^*=(V,E), \sigma,\tau\in V$, with capacities $c^*:E\to\Re^+$ with $c^*(e)=c(e)+42$ for all $e\in E$, cut (S,T) must be a minimum cut.

5. (8 points) Suppose that we are given
   - network $G=(V,E)$,
   - $\sigma,\tau\in V$,
   - $c:E\to\mathbb{Z}^+$,
   - maximum flow f in G,
   - edge $e\in E$.

We want to compute the maximum flow f^* in $G^*=(V,E)$, $\sigma,\tau\in V$, $c^*:E\to\Re^+$ where

$$c^*(\varepsilon)=\begin{cases} c(\varepsilon), \text{if } \varepsilon\neq e \\ c(\varepsilon)+1, \text{if } \varepsilon=e \end{cases}.$$

That is, G^* is identical to G except that the capacity of e is increased by 1. Find an algorithm to compute f^* in time in $O(|V|+|E|)$.

6. (6 points) Adapted from *Algorithm Design* by Kleinberg & Tardos.
As a response to natural disasters, we want an algorithm to assign n injured people to k hospitals. There is an unlimited supply of ambulances, but everybody should be transported to a hospital within a 30 minute drive of their current location. Finally, in order to balance the load, every hospital should receive at most $\lceil n/k \rceil$ people. Describe a polynomial time algorithm to accept as input an $n \times k$ array of driving times, and which will decide if a balanced assignment of people to hospitals is possible.

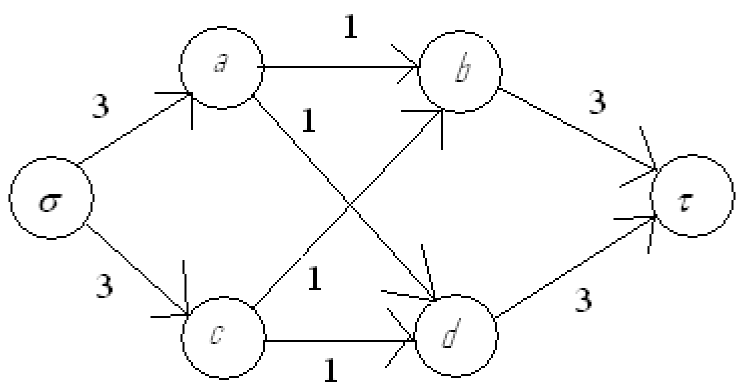
CS584
HW#6 SOLUTIONS

1. f_1+f_2 satisfies **Skew Symmetry** and **Flow Conservation**, but might violate the **Capacity Constraint**.

2. Assign a capacity of 1 to each edge $e \in E$.. Run the Ford-Fulkerson Labelling Algorithm to compute the maximal flow from x to y. Each augmenting path from $x \rightarrow y$ is edge disjoint from all previous augmenting paths. The algorithm should respond `yes` if there's a flow of at least k from $x \rightarrow y$, and `no` otherwise.

3. For each pair of vertices u, v, of G, define a flow network G_{uv} with $s=u$, $t=v$, and all edge capacities 1. The maximum flow in G_{uv} yields a minimum, and hence a minimum set of edges which separate u and v. The minimum over all pairs u and v is the edge connectivity of G.

4. The CONJECTURE is false. Network



has minimum cut $(\{\sigma, a, c\}, \{b, d, \tau\})$. But the network obtained when 42 is added to every capacity has two min cuts, $(\{\sigma\}, \{a, b, c, d, \tau\})$ and $(\{\sigma, a, b, c, d\}, \{\tau\})$.

5. We form a graph $\Upsilon(G^*)$ with edges of positive residual capacity relative to f. That is, $\Upsilon(G^*) = (V, E')$ where $E' = \{\varepsilon \in E \mid c^*(\varepsilon) - f(\varepsilon) > 0\}$. Each $\varepsilon \in E'$ has capacity $\kappa(\varepsilon) = c^*(\varepsilon) - f(\varepsilon)$. That is, the capacity of ε is its residual capacity relative to f. Constructing $\Upsilon(G^*)$ takes time in $O(|V|+|E|)$. We seek a path in $O(|V|+|E|)$ from σ to τ. If such a path exists, it is an augmenting path relative to f, and it can be found in time in $O(|V|+|E|)$.

6. Let the hospitals be $\{h_1, ..., h_k\}$ and the people be $\{p_1, ..., p_n\}$. We form a network with vertices $\{h_1, ..., h_k, p_1, ..., p_n, \sigma, \tau\}$ and edges $\sigma p_i, 1 \le i \le n$, all of capacity 1, and edges

$h_j\tau, 1 \le j \le k$, all of capacity $\lceil n/k \rceil$. Finally, there is an edge $p_i h_j, 1 \le i \le n, 1 \le j \le k$ of capacity 1 if and only if p_i is within a 30 minute drive of h_j. There exists a balanced assignment of people to hospitals if and only if the network admits a flow of volume *n*.