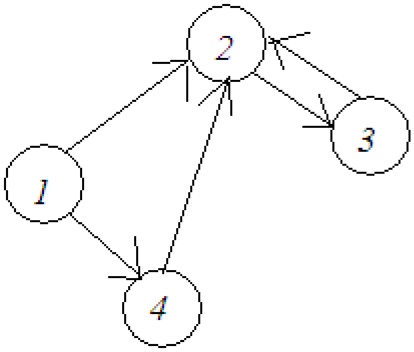
CS584
HW#5

DUE: Monday, November 30

1. (10 points) We want an algorithm to accept as input an adjacency array *E* of a graph,

$$E\left[i,j\right]=\begin{cases}1,\text{ if } v_i v_j \in E\\ 0,\text{ if } v_i v_j \notin E\end{cases}$$

and compute array NUMPATHS, where NUMPATHS[*i*,*j*] is the number of paths from v_i to v_j. For



your algorithm should return NUMPATHS:

$$\begin{array}{c|cccc} & v_i & v_2 & v_3 & v_4 \\ v_1 & 1 & \infty & \infty & 1 \\ v_2 & 0 & \infty & \infty & 0 \\ v_3 & 0 & \infty & \infty & 0 \\ v_4 & 0 & \infty & \infty & 1 \end{array}$$

2. (15 points) Professor K has *n* sensors distributed in space. Each sensor can communicate with some of its near neighbors. Professor K wants to distribute *k* transmitters to *k* of the sensors in order to broadcast information on the state of the sensors in its neighborhood back to the base station. A sensor can transmit its state to the base station if it has a transmitter or there is a sensor with a transmitter which is sufficiently close. The problem is that Professor K is cheap, and wants to minimize *k*, while assuring that all the sensors can communicate with the base station through a transmitter.

In trying to design an algorithm for Professor K, who is clueless in algorithmics, we make a simplifying assumption: each sensor can communicate with δ of its neighbors, and the δ neighbors are chosen at random from the *n* sensors (we ignore the geometry of the space in which the sensors are distributed). That is, we assume that each sensor can communicate with exactly δ of its neighbors.

Later in this course, we will see that finding an optimal value of *k* is NP-hard. Now we study a randomized algorithm to approximate an optimal value of *k*.

a What is the smallest possible value of *k* which will satisfy Professor K's constraints? Justify your answer.

b If we randomly assign *k* transmitters to *k* sensors (chosen from a uniform distribution, without replacement, over all the sensors), then what is the expected number of sensors which can not transmit back to the base station?

c Under the assumption of ***b***, what fraction of the sensors can be expected to be able to transmit to the base station? Your answer does not have to be in closed form.

3. (4 points) Give a $O\left(n^3\right)$ algorithm to count the number of triangles in a graph.

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SOLUTIONS

1. We let $NumPaths_k[i,j]$ contain the number of paths from v_i to v_j of length exactly k, and we note that $NumPaths_0 = I$, and $NumPaths_1 = E$. For any $k \geq 2$, we note that

$$NumPaths_{k+1}[i,j] = \sum_{\substack{l \in V \\ lj \in E}} NumPaths_k[i,l]$$

In order to avoid multiple counting of paths, this recurrence counts paths from v_i to v_j with a next to last vertex v_l. This is the same as $NumPaths_{k+1} = NumPaths_k * E$, which, with the initial conditions above, yields $NumPaths_k = E^k$. So for all paths with no cycle on the path, NumPaths $= I + \sum_{1 \leq k \leq n} E^k$. This is correct for all finite values of NumPaths.

To determine the values of NumPaths[i,j] which should be infinite, we seek vertices v_k which lie on a cycle and such that there exist paths from v_i to v_k and then from v_k to v_j.

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$NumPaths_n \leftarrow I + \sum_{1 \leq k \leq n} E^k$
for $i \leftarrow 1$ to $n$ do
   for $j \leftarrow 1$ to $n$ do
      for $k \leftarrow 1$ to $n$ do
         if $(NumPaths_n[i,k] > 0) \wedge (NumPaths_n[k,k] > 1) \wedge (NumPaths_n[k,j] > 0)$
         then $NumPaths_n[i,j] \leftarrow \infty$
return $NumPaths_n$
```

2. This problem is well modeled as a graph of n vertices/sensors, where two vertices are adjacent (there is an edge between them) if the sensors can communicate. So each vertex has degree δ.

a Each transmitter can "cover" at most k+1 sensors, itself and k neighbors. In the best case we need $\left\lceil \frac{n}{\delta+1} \right\rceil$ transmitters. This happens when the sensors are clustered into disjoint complete graphs.

b The probability that a random $v \in V$ is covered by a random transmitter is $\frac{\delta+1}{n}$, so the probability that it is not covered is $1-\frac{\delta+1}{n}$. And since the transmitters are distributed independently, the probability that v is not covered by any of the transmitters (that can not transmit back to the base station) is $\left(1-\frac{\delta+1}{n}\right)^k$. By linearity of expectation, the expected number of sensors which cannot transmit to the base station is $n\left(1-\frac{\delta+1}{n}\right)^k$

c We define random variable $X_i, 1 \le i \le n$ to be 1 if sensor *i* can transmit to the base station, and 0 otherwise. So

$$E[X_i] = 1*\left(1-\left(1-\frac{\delta+1}{n}\right)^k\right) + 0*\left(1-\frac{\delta+1}{n}\right)^k = \left(1-\left(1-\frac{\delta+1}{n}\right)^k\right)$$

Letting random variable *X* denote the number of sensors which can transmit to the base station, $X = \sum_{1\le i\le n} X_i$, and, by linearity of expectation,

$$E[X] = E\left[\sum_{1\le i\le n} X_i\right] = \sum_{1\le i\le n} E[X_i] = n\left(1-\left(1-\frac{\delta+1}{n}\right)^k\right).$$

So the expected fraction of transmitting sensors is

$$\frac{n\left(1-\left(1-\frac{\delta+1}{n}\right)^k\right)}{n} = 1-\left(1-\frac{\delta+1}{n}\right)^k.$$

3.
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$A \leftarrow E*E*E$                                             $\Theta(n^3)$
$NumTriangles \leftarrow 0$                                      $O(1)$
for $i \leftarrow 1$ to $n$ do
    $NumTriangles \leftarrow NumTriangles + A[i,i]$              $\Theta(n)$
return $NumTriangles/3$                                          $O(1)$
```

We note that each triangle contributes three times to a diagonal value.