CS524

FINAL EXAM

Name_______________

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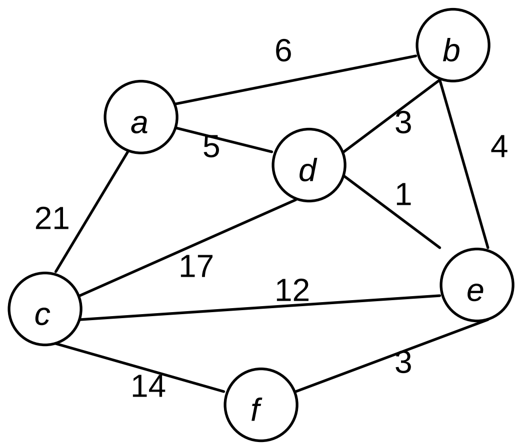
All documentation permitted

1. (20 points) Suppose that a network (*V*,*E*) has a pair of nodes, $u, v \in V$, such that $(u,v),(v,u) \in E$. That is, there is an edge from *u* to *v* **and** an edge from *v* to *u*. Prove or give a counterexample to the following:

CONJECTURE: There must be a maximum flow *f* such that $f(u,v) = 0$ or $f(v,u) = 0$.

2. (20 points) Suppose that you are given a graph G=(V,E) and a function $w: E \rightarrow \Re^+$ which associates a positive weight with every edge. We define the *distance* $D: V \times V \rightarrow \Re^+$ between two vertices as the length of a shortest path between the vertices, and the *diameter* of G as the distance between two most distant vertices. That is, $diameter(G) = \max_{v,w \in V}(D[v,w])$. For example, the diameter of



is 18 because the distance between a and c is 18. Describe an algorithm to determine the diameter of an arbitrary graph G with worst-case time complexity in $O(|V|^3)$.

3. (20 points) An instance of the THREESET problem is positive integers $x_1,\ldots,x_n$, and an integer K. An algorithm to solve the THREESET problem should return true if $x_1,\ldots,x_n$ can be partitioned into multisets S_1, S_2 and S_3 such that $\sum_{x\in S_1} x = \sum_{x\in S_2} x = \sum_{x\in S_3} x = K$ and false otherwise. That is, when presented with K=9 and integers 1,2,2,2,4,4,5,7, the algorithm should return true because of the partition into multisets {4,5}, {2,7} and {1,2,2,4}, and when presented with n=8, K=9, and integers 1,2,2,2,4,4,5,77, the algorithm should return false. Your agorithm should work in time in $O(nK)$ under the standard computational model.

4. (20 points) Suppose you are given an infinite supply of coins of denominations $\{c_1, \ldots, c_n\}$.

For example, in the US (ignoring silver dollars) n=5 and $c_1 = 1, c_2 = 5, c_3 = 10, c_4 = 25$ and c_5=50. Design an algorithm to compute the number of ways to make change for a¢. For example, for input

$$n = 5, \{1, 5, 10, 25, 50\}, a=17$$

the output would be 6 because of the solutions

$$(10,5,1,1), (10,1,1,1,1,1,1,1), (5,5,5,1,1), (5,5,1,1,1,1,1,1,1),$$
$$(5,1,1,1,1,1,1,1,1,1,1,1,1), (1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$$

5. (20 points) A recursive definition of the binomial coefficients is that

$$\binom{n}{k} = \begin{cases} 1, \text{ if } k = 0 \text{ or } k = n \\ \binom{n-1}{k-1} + \binom{n-1}{k}, \text{ otherwise} \end{cases}$$

The corresponding recursive program is

$C(n,k)$

if $k = 0$ or $k = n$

then return 1

else return $C(n-1, k-1) + C(n-1, k)$

a What is the time to execute $C(n,k)$, as a function of n and k?

b Write a dynamic programming algorithm to compute $C(n,k)$ with an execution time in $\Theta(nk)$.

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Solutions to Final Exam

1. The CONJECTURE is true. For any flow f, the value of $|f|$ doesn't change if we increase or decrease $f(u,v)$ and $f(v,u)$ by the same amount (while respecting capacity constraints). Let f be any maximum flow. If $f(u,v)>0$ and $f(v,u)>0$, then we create a new flow $f^*(x,y)=\begin{cases} f(x,y), \text{ if } \{x,y\}\neq\{u,v\} \\ f(x,y)-\min\left(f(u,v), f(v,u)\right), \text{ if } \{x,y\}=\{u,v\} \end{cases}$

That is, f^* is the same as f except through the edges $(u,v),(v,u)$, and in that case they are reduced by the same amount. And $|f|=|f^*|$.

2. Apply Dijkstra's algorithm $|V|$ times, once for each $v\in V$ as the source.

diameter $\leftarrow -\infty$
for each $v\in V$ **do**
  Apply Dijkstra's algorithm with source v to construct $D_v: V\rightarrow \Re^+$
  for each $w\in V$ **do** *diameter* $\leftarrow \max\left(\mathit{diameter}, D_v[w]\right)$

3. Given an instance of THREESET, you can call KNAPSACK with capacity K such that the value and weight of object i is x_i for $1\le i\le n$. Using the dynamic programming algorithm to solve KNAPSACK, if there is no packing of value K, then THREESET should return *false*. If there is a packing of value K, then remove these elements from the set of possible objects. Invoking the knapsack algorithm again, the remaining objects admit a packing of value K if and only if the answer to THREESET is *true*.

4. For $0\le k\le n,\ 0\le x\le a$, we let $f(k,x)$ denote the number of ways to make change for x¢ using coins of denominations $\{c_1,\ldots,c_k\}$. We assume that $f(k,x)=0$ if $k<0$ or $x<0$. We note that we can use $0,1,\ldots,\lfloor x/c_k\rfloor$ coins of denomination c_k. If we use exactly i coins of denomination c_k, then there are $f(k-1,x-ic_k)$ ways to make change for $(x-ic_k)$¢ using coins of denominations $\{c_1,\ldots,c_{k-1}\}$.

$$f(k,x)=\sum_{0\le i\le\lfloor x/c_k\rfloor} f(k-1,x-ic_k)$$

for $x\leftarrow 1$ **to** a **do**
  for $k\leftarrow 0$ **to** n **do**
    $f(k,x)\leftarrow f(k-1,x)$ **/* using 0 c_k */**
    for $i\leftarrow 1$ **to** $\lfloor x/c_k\rfloor$ **do**

$$f(k,x) \leftarrow f(k,x) + f(k-1, x - ic_k) \qquad \textbf{/* using } \boldsymbol{i\, c_k} \textbf{ */}$$

return $f(n,a)$

5. ***a*** Because ultimately the recursion tree causes $\binom{n}{k}$ 1's to be added together, computing $C(n,k)$ makes $\binom{n}{k}$ calls on C, and has a time complexity $\Theta\left(\binom{n}{k}\right)$.

b The values are stored in an array $C[0..n, 0..k]$.

for $i \leftarrow 0$ **to** n **do**

  for $j \leftarrow 0$ **to** $\min(i,k)$ **do**

    if $k = 0$ or $k = n$

    then $C[i,j] \leftarrow 1$

    else $C[i,j] \leftarrow C[i-1, j-1] + C[i-1, j]$