CS524
HW#6

DUE: Wednesday, December 15

1. (10 points) (From Lawler's *Combinatorial Optimization, Networks and Matroids*)
A *most valuable edge* of a network G is an edge whose removal from G reduces the maximum flow as much as the deletion of any other edge. Prove or give a counterexample to each of the following conjectures.

CONJECTURE 1: If (x,y) is a most valuable edge of G, then there must be a minimum cut (S,T) of G with $x \in S$ and $y \in T$.

CONJECTURE 2: For any minimum cut (S,T) of G and any edge (x,y) with $x \in S$ and $y \in T$ with maximum capacity in (S,T), (x,y) must be a most valuable edge of G.

2. (8 points) Suppose there are n men, n women and m marriage brokers. Some pairs of men and women are willing to marry each other, and some are not. Each broker has either a list of some of the men or a list of some of the women as clients, and can arrange a marriage for any person on the list. A person may have more than one broker. Finally, we restrict the maximum number of marriages that broker i can arrange to b_i. All marriages are heterosexual, all men and women are monogomous, and any person who marries must have a broker. Describe an efficient algorithm to compute the maximum number of marriages that can be arranged.

3. (8 points) Do **Exercise *34.1-1*** on page 978 of our text.

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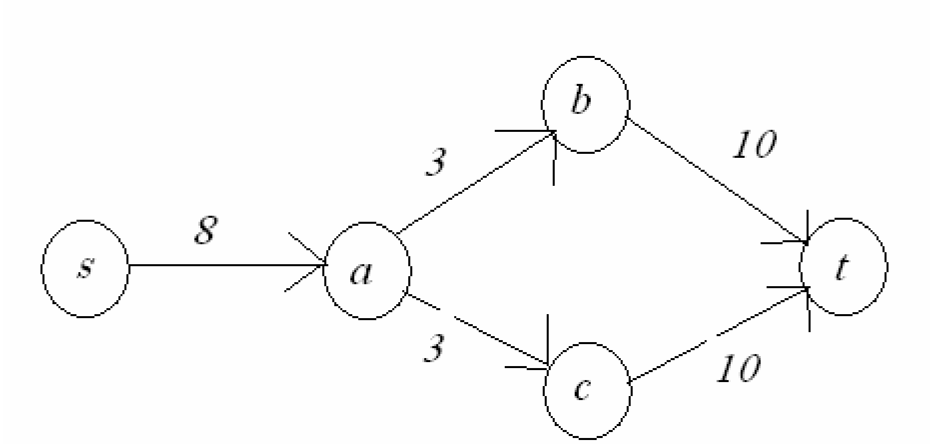
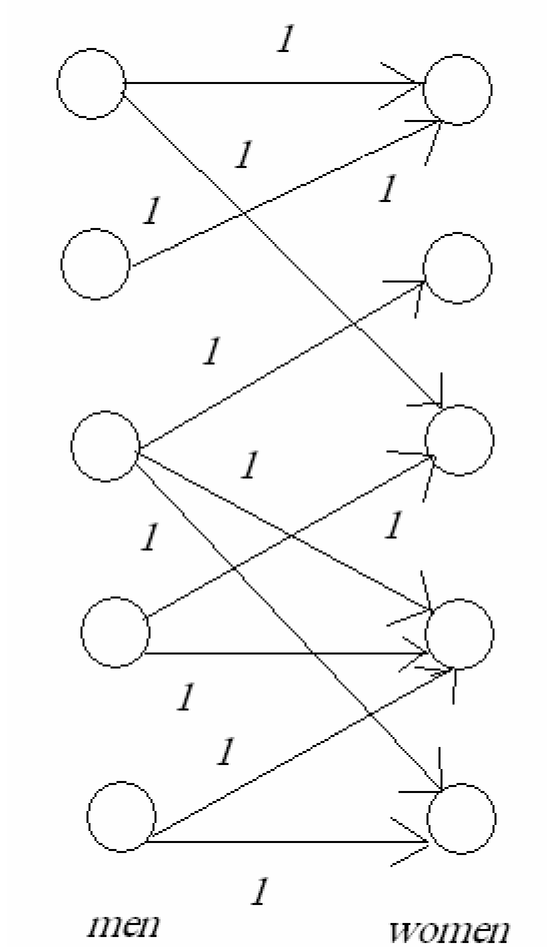
HW#6 SOLUTIONS

1. CONJECTURE 1 and CONJECTURE 2 are both false. The network

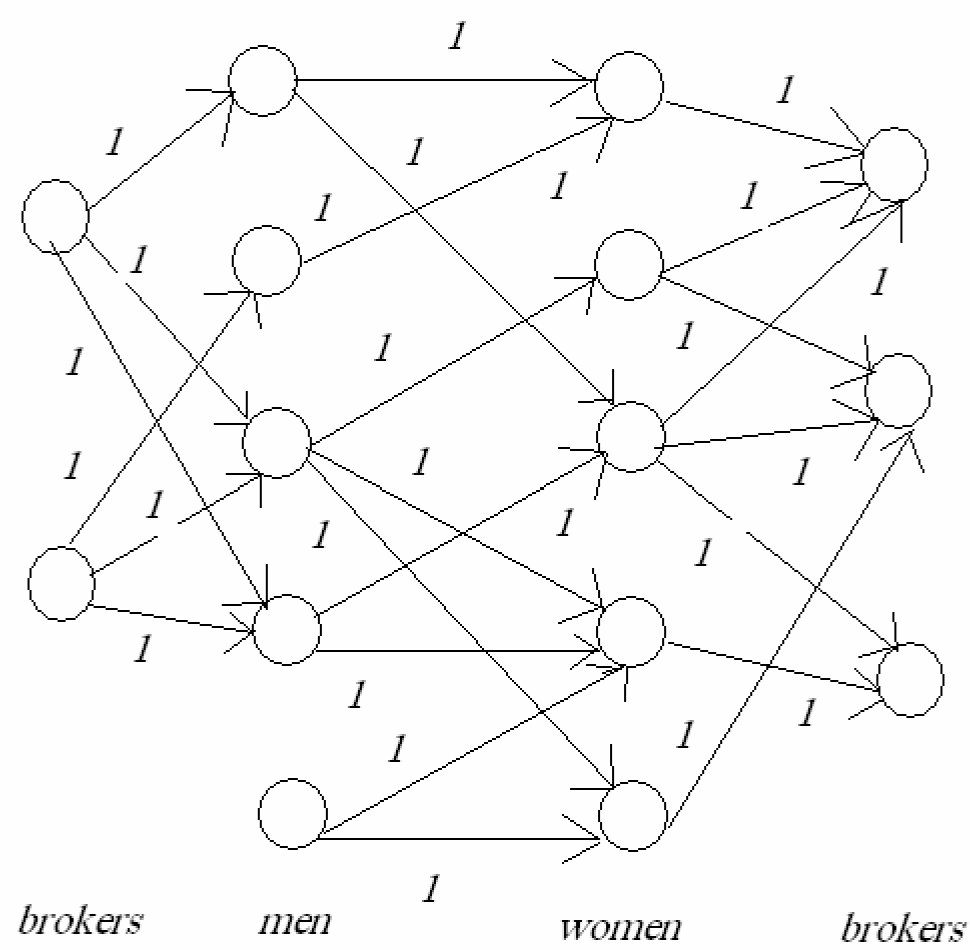


admits exactly one minimum cut, $(\{s,a\},\{b,c,t\})$, and exactly one most valuable edge, (s,a). But (s,a) does not belong to $(\{s,a\},\{b,c,t\})$.

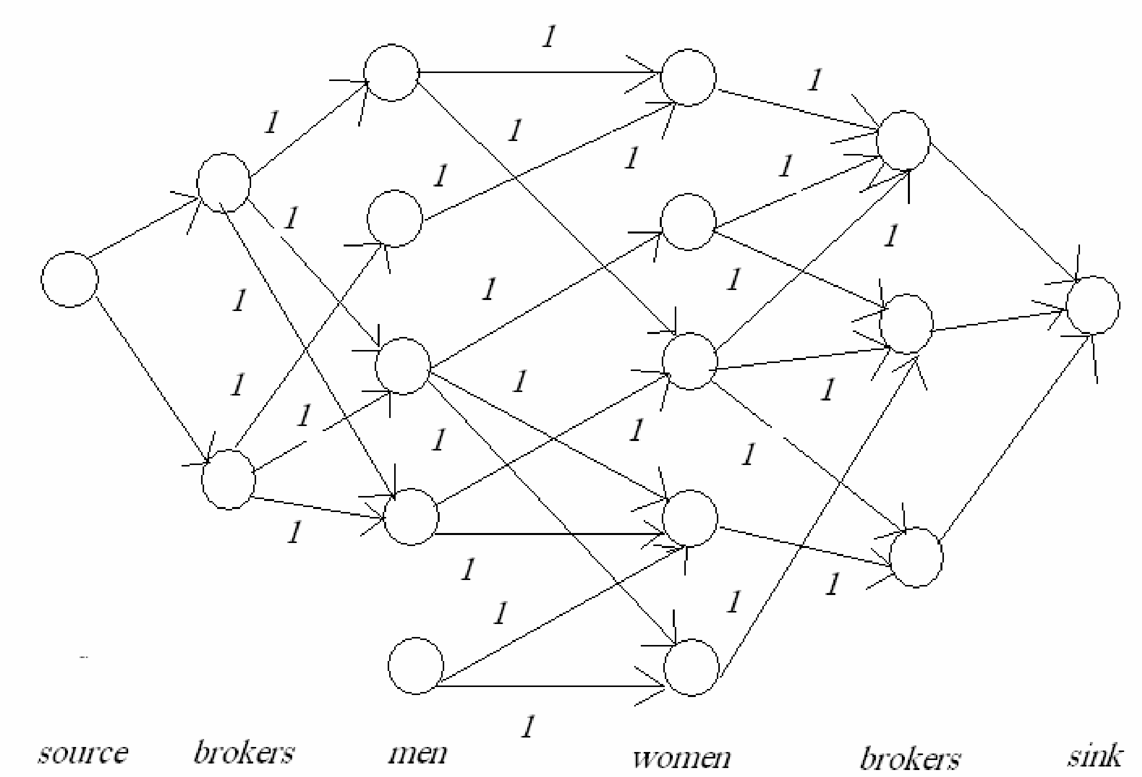
2. We start the network as a bipartite graph, with the n men in one class and the n women in the other, and there is an edge from a man to a woman if and only if they are willing to marry each other. We assign a capacity of 1 to each edge.



For each agent *i* who has men as clients, there is an edge of capacity 1 from vertex *i* to *i*'s clients. For each agent *i* who has women as clients, there is an edge of capacity 1 from *i*'s clients to vertex *i*.


1
brokers
men
women
brokers

Finally we add a source with an edge of capacity b_i to broker i with male clients, and a sink with an edge of capacity b_j from each broker j with female clients.



The edges with flow 1 in a maximum flow in the network correspond to a maximum matching. We note that starting with a flow of 0, the Ford-Fulkerson Algorithm maintains integral flows though every edge. A man or a woman can not be married unless he or she has a flow of 1 through one of his or her brokers, and broker i can not be involved in more than b_i marriages.

3. If LONGEST-PATH-LENGTH.could be solved in polynomial time, then we could solve the decision problem LONGEST-PATH in polynomial time for instance $\langle G,u,v,k\rangle$ by running LONGEST-PATH-LENGTH on $\langle G,u,v\rangle$ and returning true if and only if LONGEST-PATH-LENGTH returned a value $\geq k$.

If LONGEST-PATH $\in P$, then we identify the length of the longest path by essetially doing a sequential search (binary search would be faster, though still in P) over all possible path lengths.

LONGEST-PATH(G,u,v)
 $k \leftarrow 0$
 while LONGEST-PATH(G,u,v,k+1) **do** $k \leftarrow k+1$
 return k