CS524
HW#3

DUE: Monday, February 28

1. (3 points) Do **EXERCISE *7.2-6*** on pg. 153 of our text.

2. (4 points) Do **EXERCISE *7.4-6*** on pg. 159 of our text.

3. (10 points) MEDIAN-OF-3 QUICKSORT. Write a program QUICKSORT, either from the algorithm on pg. 146 of the text or from class or from the CD that came with the text or from elsewhere. Time your program for a large, random array *A*. Then modify your program to implement MEDIAN-OF-3 QUICKSORT (described in class and in **PROBLEM *7-5*** on pg. 162 of our text) and time the new program. What % speed-up do you get from your program?

4. (7 points) (From Baase and Van Gelder's *Computer Algorithms*) Suppose you have an algorithm $\mathcal{A}_1$ to solve a problem of size *n* in $\Theta(n^2)$ steps. You discover a divide-&-conquer algorithm $\mathcal{A}_2$ which can, in $\Theta(n \lg n)$ time, divide a problem of size *n* into two subproblems of size *n*/2, and then, in time $\Theta(n \lg n)$, recombine two solutions of size *n*/2 into a solution of the original problem. Which algorithm, $\mathcal{A}_1$ or $\mathcal{A}_2$, is more efficient? You may assume that *n* is a power of 2. Justify your response.

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HW#3 SOLUTIONS

1. The array $A[1..n]$ is PARTITIONed into subarrays of sizes αn and $(1-\alpha)n$ if the pivot element is the α^{th} smallest or the α^{th} largest element of *A*. If the pivot element lies between these elements (the α^{th} smallest and the α^{th} largest), then the split is at least as balanced as $1-\alpha$ to α. Since $1-2\alpha$ elements lie within this window, the probability of a split at least as balanced as α to $(1-\alpha)$ is approximately $1-2\alpha$.

2. The only way the split could be worse than a α-to-$(1-\alpha)$ is if two of the three numbers are among the αn smallest or the αn largest elements of *A*. The probability that exactly two of the numbers are among the αn smallest is $\binom{3}{2}\alpha^2(1-\alpha)=3\alpha^2(1-\alpha)$,

The probability that all three of the numbers are among the αn smallest is α^3. The probability that at least two of the numbers are among the αn smallest is $3\alpha^2(1-\alpha)+\alpha^3=\alpha^2(3-2\alpha)$. The events of the two smallest being among the αn smallest and the two largest being among the αn largest are mutually exclusive, so the probability of the union of these events is the sum of the individual proababilities. So the probability of the two smallest being among the αn smallest and the two largest being among the αn largest is $2\alpha^2(3-2\alpha)$, which is the probability of getting worse than a α-to-$(1-\alpha)$ split.

4. Letting $T(n)$ denote the time for $\mathcal{A}_2$ to solve the problem, we derive the recurrence

$$T(n)=2T(n/2)+cn\lg n$$

where *c* is the sum of the constants hidden in the asymptotic notation for the complexities of the division and the recombination of $\mathcal{A}_2$. Unfolding the recurrence,

$$T(n)=2T(n/2)+cn\lg n$$

$$=2\left(2T(n/4)+c\frac{n}{2}\lg\frac{n}{2}\right)+cn\lg n=4T(n/4)+cn\lg n+cn(\lg n-1)$$

$$=4\left(2T(n/8)+c\frac{n}{4}\lg\frac{n}{4}\right)+cn\lg n+cn(\lg n-1)=8T(n/8)+cn\lg n+cn(\lg n-1)+cn\lg(n-2)$$

After *k* levels of unfolding,

$$T(n)=2^kT(n/2^k)+cn\left(\lg^2 n-(1+2+...+(k-1))\right)$$

$$=2^kT(n/2^k)+cn\left(\lg^2 n-\frac{k(k-1)}{2}\right)$$

The unfolding stops when $2^k = n$, or k=lg$n$. So

$$T(n) = 2^{\lg n} T\left(n/2^{\lg n}\right) + cn\left(\lg^2 n - \frac{\lg n(\lg n - 1)}{2}\right) = n + cn\left(\lg^2 n - \frac{\lg n(\lg n - 1)}{2}\right) \in \Theta\left(n \lg^2 n\right)$$

This also follows from the version of the Master Theorem we did in class, though not the text's version.