CS524
HW#1

DUE: Monday, January 31

1. (2 points) Do **Exercise *C.3-3*** on pg. 1111 of our text.

2. (16 points) Consider the following program fragment operating on array $A[n]$:

```
max ← -∞;
for i ← 1 to n do
   if A[i]>max then max←A[i]
```

Assume that A contains a permutation of $\{1,\ldots,n\}$ drawn from a uniform distribution over the set of all such permutations. Give a closed form for each of the following answers.

a What is the expected number of times $max \leftarrow A[i]$ is executed?

b What is the variance of the number of times $max \leftarrow A[i]$ is executed?

c What is the standard deviation of the number of times $max \leftarrow A[i]$ is executed?

d What is the probability that $max \leftarrow A[i]$ is executed n times?

e For n=10, what is the probability that $max \leftarrow A[i]$ is executed 10 times?

f For n=10, use Chebyshev's inequality to bound the probability that $max \leftarrow A[i]$ is executed n times?

3. (5 points) Do **Exercise *5.2-5*** on pg. 99 of our text.

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SOLUTION FOR H.W. #1

1. The player's expected gain is $-\$1*\Pr\{\$0\}+\$1*\Pr\{\$1\}+\$2*\Pr\{\$2\}+\$3*\Pr\{\$3\}$. For any die, the probability that it's the player's choice is 1/6, and the probability that it isn't is 5/6 . The probability that exactly k dice, $0 \le k \le 3$, are the player's choice is $\binom{3}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{3-k}$. So the player's expected gain is

$$-1\left(\frac{5}{6}\right)^3+\sum_{1\le k\le 3} k\binom{3}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{3-k} = \text{-0.07870370370}.$$ Playing the game is expensive.

2. ***a*** Define n random variables, $X_i = \begin{cases} 1, \text{if } \ max = A[\,i\,] \text{ is executed} \\ 0, \text{if } \ max = A[\,i\,] \text{ is not executed} \end{cases}$, for $n \ge i \ge 1$, and define random variable $X = \sum_{n\ge i\ge 1} X_i$ which counts the number of times max=A[i] is executed. Since Pr{ max=A[i] is executed}=1/i, the expected value of X_i is 1/i, and

$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{n\ge i\ge 1} X_i\right] = \sum_{n\ge i\ge 1}\mathrm{E}[X_i] = \sum_{n\ge i\ge 1}\frac{1}{i} = H_n .$$

b Noting that X_i and Xj are independent for $i \neq j$, it follows that $VX = V\sum_{n\ge i\ge 1} X_i = \sum_{n\ge i\ge 1} VX_i$. To compute VX_i, we note that $\mathrm{E}[X_i^{\,2}] = 1^2 * \Pr\{X_i = 1\} = 1/i$, and

$$VX_i = \mathrm{E}[X_i^{\,2}] - (\mathrm{E}[X_i])^2 = \frac{1}{i} - \frac{1}{i^2}, \text{ so that } VX = \sum_{n\ge i\ge 1} VX_i = \sum_{n\ge i\ge 1}\left(\frac{1}{i}-\frac{1}{i^2}\right) = H_n - H_n^{(2)}.$$

c The standard deviation of X is $\sqrt{H_n - H_n^{(2)}}$.

d X= n if A is in increasing order. The probability of this event is 1/n!.

e 1/10!=.000000275573

f H_{10}=2.928968 and $H_{10}^{(2)}$=1.5497674, so VX=1.3792006 and the standard deviation of X is 1.174393716. For n=10, E[X]= 2.928968, so |X-μ|=7.071032, which is 6.02 standard deviations from the mean of 2.928968 Chebyshev's inequality states that the likelihood of this event is less than or equal to .02759, a weak bound indeed.

3. For each pair $1 \le i < j \le n$, define the indicator random variable

$$X_{i,j} = \begin{cases} 1, \text{if } \ A[i] > A[j] \\ 0, \text{if } \ A[i] \le A[j] \end{cases}$$

For all $1 \le i < j \le n$,

$$E\left[X_{i,j}\right] = 1 * \Pr\{A[i] > A[j]\} + 0 * \Pr\{A[i] \le A[j]\} = 1/2$$

So $E\left[\sum_{1 \le i < j \le n} X_{i,j}\right] = \sum_{1 \le i < j \le n} E\left[X_{i,j}\right] = \sum_{1 \le i < j \le n} 1/2 = \frac{1}{2}\frac{n(n-1)}{2} = \frac{n(n-1)}{4}$