CS524
HW#1

DUE: Wednesday, September 22

1. (16 points) Assume there is a **fair** casino in which for any $n \in \mathbb{Z}^+$, if you bet \$*n*, you win with probability ½, and if you win the casino gives you \$2*n*, and if you lose the casino gives you \$0. The probability of winning any game is independent of the probability of winning any other game.
(*a*) In the *Martingale* system of betting, you first bet \$1, and then, if you lose, you bet \$2, and then, if you lose, you bet \$4... You quit as soon as you win. In general, at round *k*, $k \geq 1$, you bet $\$2^{k-1}$, and you quit the game as soon as you win, and you have an unlimited amount of money. What is the expected number of games you will play?
(*b*) What is your expected amount of *winnings* when you leave the casino? Your *winnings* are the amount of money you won minus the amount of money you *invest*ed in playing the game. Note that according to this definition, *winnings* can be negative. Prove your answer (it does not suffice to detect a pattern for several cases and then infer the answer).
(*c*) Assume that a fair casino exists. Does Martingale "guarantee" that you will win money. Why couldn't you use the system to guarantee an income? That is, every time you need money you'd walk into the casino and apply the Martingale.
(*d*) What is the expected amount of money you *invest* in the game until you win? That is, if you win the first game, then you invested \$1. If you win the third game, then you *invest*ed \$7.

2. (5 points) What is the maximum number of regions of the plane defined by $n \geq 0$ (infinite) straight lines? To start things rolling, we note that for *n*=0 there is 1 region, *n*=1 line partitions the plane into 2 regions, and *n*=2 lines divide the plane into 4 regions. Give a closed form for your answer.

3. (6 points) Given an array $A[1..n]$ of numbers, we seek to compute an array $Aver[1..n]$ such that for each k, $1 \le k \le n$, element $Aver[k]$ is the average of $A[1],...,A[k]$. An algorithm to solve this problem is

```
for $k \leftarrow 1$ to $n$ do
    $sum \leftarrow 0$
    for $i \leftarrow 1$ to $k$ do
        $sum \leftarrow sum + A[i]$
    $Aver[k] \leftarrow sum / k$
```

(*a*) How many times is the instruction $sum \leftarrow sum + A[i]$ executed? Give an exact answer (don't use asymptotic notation).
(*b*) Give a linear time algorithm to solve this problem.

4. (6 points) Assuming that n is a power of 2, what value is returned by the following algorithm? Give an exact answer; do not use asymptotic notation.

```
$k \leftarrow n$
$sum \leftarrow 0$
while $k \ge 1$ do
    for $i \leftarrow 1$ to $k$ do
        $sum \leftarrow sum + 1$
    $k \leftarrow k / 2$
return $sum$
```

CS524
HW#1 SOLUTIONS

1. (*a*) We let rv *X* denote the number of times you play until you win, including the time you win. We are asking for $E[X]$. We note that *X* is geometrically distributed, with probability of success $p=½$. $E[X]=\sum_{k\ge1} k*\Pr\{X=k\}$. The only way for $X=k$ is to lose k-1 times (with probability $(1/2)^{k-1}$) and then succeed (with probability ½). Thus, $E[X]=\sum_{k\ge1} k*(1/2)^{k-1}(1/2)=\sum_{k\ge1} k*\left(\frac{1}{2}\right)^k$. By Equation C.31 of our text, $E[X]=\frac{1}{1/2}=2$.

(*b*) With probability ½ you walk away with *winnings* of \$1 after one play. With probability ¼ you lose the first game (costing you \$1) and then win the second game (costing you \$2), and then walk away with a *winnings* of \$1. In general, for the event that you walk away after *k* games, $k\ge1$, then it cost you $\$\left(2^k-1\right)$, and you win $\$2^k$. The probability of this event is $(1/2)^k$. So your expected *winnings* are $\sum_{k\ge1}\left(2^k-\left(2^k-1\right)\right)(1/2)^k=\sum_{k\ge1}(1/2)^k=\sum_{k\ge0}(1/2)^k-1=1$, where the last equality follows from Equation A.6 for the sum of an infinite geometric series.

(*c*) The Martingale guarantees you \$1 assuming that you walk in with an infinite amount of money.

(*d*) Looking at the solution to part (*b*), if you win game *k*, then your investment is $\$\left(2^k-1\right)$, and your total expected investment is

$$\sum_{k\ge1}\left(2^k-1\right)(1/2)^k=\sum_{k\ge1}2^k(1/2)^k-\sum_{k\ge1}(1/2)^k=\left(\sum_{k\ge1}1\right)-1$$

which is unbounded. This shows why you can't afford to play the Martingale.

2. We let x_n denote the answer. The problem statement shows that $x_0=1,\ x_1=2$ and $x_2=4$. For $n\ge1$, the n^{th} line divides the first region it traverses, and each time it crosses one of the previous *n*-1 lines, it starts dividing the next region. Hence, x_n is characterized by the recurrence

$$x_n=\begin{cases}x_{n-1}+n, \text{ for } n\ge1\\ 1, \text{ for } n=0\end{cases}=1+\begin{cases}x_{n-1}+n, \text{ for } n\ge1\\ 0, \text{ for } n=0\end{cases}$$

By Equation A.1, the closed form solution for this arithmetic series is

$$x_n=\sum_{n\ge k\ge0}k=\frac{n(n+1)}{2}+1\ .$$

3 (*a*) Converting the program directly to a summation, $\sum_{1\le k\le n}\sum_{1\le i\le k}1=\sum_{1\le k\le n}k=\frac{n(n+1)}{2}$.

(*b*) In computing $sum=\sum_{1\le i\le k}A[i]$, we note that $\sum_{1\le i\le k-1}A[i]$ has already been computed.

```
 $sum \leftarrow 0$
**for** $k \leftarrow 1$ **to** $n$ **do**
        $sum \leftarrow sum + A[k]$
        $Aver[k] \leftarrow sum / k$
```

4. After the 1st time through the **for**-loop, when *k*=*n*, *sum* is set to *n*. After the 2nd time through the **for**-loop, when *k*=*n*/2, *sum* is set to 3*n*/2. After the *j*th time through the loop, when $k=\frac{n}{2^{j-1}}$, *sum* is $n+\frac{n}{2}+...+\frac{n}{2^{j-1}}$. This summation can be rewritten as

$$\sum_{1\le i\le j}\frac{n}{2^{i-1}}=n\sum_{1\le i\le j}\left(\frac{1}{2}\right)^{i-1}\quad .$$

Replacing *i* by *i*+1 yields $n\sum_{1\le i+1\le j}\left(\frac{1}{2}\right)^{i}=n\sum_{0\le i\le j-1}\left(\frac{1}{2}\right)^{i}=n\frac{1-\left(\frac{1}{2}\right)^{j}}{1-\frac{1}{2}}=2n\left(1-2^{-j}\right)$.

The **while**-loop executes lg*n*+1 times, so the algorithm returns

$2n\left(1-\frac{1}{2^{\lg n+1}}\right)=2n\left(1-\frac{1}{2n}\right)=2n-1.$