CS503

FINAL EXAM

Name_____

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All documentation permitted, but no communication is permitted

1. (20 points) Which of the following are context-free? Justify your answers.

a $\left\{a^n b^n c^{3n} \mid n \geq 0\right\}$

b $\left\{a^n b^m c^{3n} \mid m, n \geq 0\right\}$

2. (30 points) Let $\mathcal{P}$ be the set of languages for which membership can be decided by aTuring machine in polynomial time, and let $\mathcal{NP}$ be the set of languages for which membership can be decided by a nondeterministic Turing machine in polynomial time.

a Is $\mathcal{P}$ closed under concatenation? Justify your answer.

b Is $\mathcal{NP}$ closed under concatenation? Justify your answer.

c Is $\mathcal{P}$ closed under complement? Justify your answer.

3. (20 points) Is the following CONJECTURE true? Justify your answer.

CONJECTURE: The set of recursively enumerable languages is closed under complement.

4. (30 points) Consider the language

$$L = \left\{ \left(M_0, M_1, z \right) \mid \text{TM } M_0 \text{ halts on input } z \in \Sigma^* \text{ in fewer steps than TM } M_1 \right\} \quad .$$

Note that if M_0 halts on z but M_1 does not, then $\left(M_0, M_1, z \right) \in L$.

a Is L recursively enummerable? Justify your answer.

b Is L recursive? Justify your answer.

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Solutions to FINAL EXAM

1. ***a*** $\{a^n b^n c^{3n} \mid n \geq 0\}$ is not a CFL. Assume it is a CFL, and let k be the constant provided by the Pumping Lemma for CFLs. Consider $z=a^k b^k c^{3k}$. By the PL, z can be written $uvwxy$ with $length(v)+length(x)>0$ and $length(vwx) \leq k$ such that $\{uv^i wx^i y \mid i \geq 0\} \subseteq \{a^n b^n c^{3n} \mid n \geq 0\}$. Because $length(vwx) \leq k$, vx can not contain a's **and** c's. But because $length(v)+length(x)>0$, string vx must contain some a's or b's or c's. So uv^2wx^2y can not increase the number of a's, b's **and** c's. So $uv^2wx^2y \notin \{a^n b^n c^{3n} \mid n \geq 0\}$, which is a contradiction.

b $\{a^n b^m c^{3n} \mid m,n \geq 0\}$ is a CFL, generated by the grammar:

$$S \rightarrow aSccc \mid B$$
$$B \rightarrow bB \mid \lambda$$

2. ***a*** & ***b*** $\mathcal{P}$ and $\mathcal{NP}$ are each closed under concatenation. Assume that $L_0, L_1 \in \mathcal{P}$ (respectively $L_0, L_1 \in \mathcal{NP}$). There must exist TMs M_0, M_1 (respectively TMs M_0, M_1) such that $L_0 = L(M_0)$ and $L_1 = L(M_1)$, and M_0 and M_1 decide on membership in L_0 and L_1 in polynomial time. We want to show that $L_0 \bullet L_1 \in \mathcal{P}$ (respectively $L_0 \bullet L_1 \in \mathcal{NP}$).

For both cases (deterministic and nondeterministic), we can test whether $z \in L_0 \bullet L_1$ by using the following algorithm:

for each $x, y \in \Sigma^*$ such that $xy=z$ **do**
  if $x \in L(M_0) \wedge y \in L(M_1)$ **then return** accept
return reject

There are $length(z)+1$ decompositions of z into x and y, so there are a polynomial (in $length(z)$) number of invocations of two polynomial time algorithms. So the test is performed in polynomial time. And the algorithm decides that $z \in L_0 \bullet L_1$ if and only if z can be decomposed into x and y such that $x \in L_0 \wedge y \in L_1$.

c $\mathcal{P}$ is closed under complement. Assume that $L \in \mathcal{P}$. There must exist TM M such that $L = L(M)$ and M decides on membership in L in polynomial time. To decide whether $z \in \overline{L}$, we run the following TM which halts in polynomial time.

run M on input z
if M **returns** accept **then return** reject **else return** accept

3. The set *HALT* of pairs (M, z) where TM M halts on input z is re but not recursive (this is equivalent to the halting problem). We proved in class, and it is proved in the text, that

if L and $\overline{L}$ are both re then L is recursive. So *HALT* is re, but its complement is not re. So the CONJECTURE is false.

4. ***a*** L is re. We can use two Universal TMs to simulate, step by step, M_0 and M_1 on input z. If M_0 halts first, then we accept. If M_1 halts first, then we reject.
b L is not recursive. The HALTING PROBLEM is reducible to membership in L, and hence membership in L can not be recursive. Let M_∞ be a TM which never halts. For example, M_∞ can have one state, q_0, and $\delta_{M_\infty}(q_0, x) = [q_0, x, R]$ for all $x \in \Gamma$. Given any instance of the HALTING PROBLEM, such as whether TM M halts on input z, we note that the answer is yes if and only if $(M, M_\infty, z) \in L$. Hence, if membership in L were recursive, then the HALTING PROBLEM would be decidable.