CS503

HW#1

DUE: Tuesday, September 22

1. (1 point) Do **Exercise 27** on page 38 of the text.

2. (16 points) Prove **Theorem 1.4.4** on pages 18 and 19 of our text.

3. (10 points) Assume that X and Y are infinite disjoint sets of nodes and consider the digraph with nodes $N = X \cup Y$. In the digraph, each node of X (respectively Y) has exactly one arc going out (it has out-degree 1) and it goes to a node of Y (respectively X). Furthermore each node has in-degree at most 1. A *component* of a digraph is a minimal set of nodes (and the arcs between them) such that every pair of nodes with a path between them belong to the same component.
i) Describe the possible components of the digraph.
ii) For each possible component, describe a bijection between the nodes from X in the component and the nodes from Y.
iii) Conclude **Theorem 1.4.2** for infinite disjoint sets.

4. (5 points) Give regular expressions for the following languages over $\Sigma = \{0,1\}$:
i) Strings of length less than 4.
ii) Strings that contain 001 as a substring.
iii) Strings with no pair of consecutive 1's.
iv) Strings whose next to last character is a 1.
v) Strings that do not begin with 00.

5. (2 points) Give a string $u \in \{0,1\}^*$ such that $length(u) = 8$ but u does not belong to the regular language $(00 \cup 11)^* (01 \cup 10)(00 \cup 11)^*$.

6. (2 points) Give a regular expression for the language of assignment statements such as $E \leftarrow 2.71828$ or $PI \leftarrow 3.14159265358979323$ An assignment statement is a nonempty string of uppercase letters, followed by a left arrow, "$\leftarrow$", followed by a nonempty string of digits without a leading 0, followed by a decimal point, ".", followed by a nonempty string of digits without a trailing 0.

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Solutions for HW#1

1. If $card(X) \equiv card(X)$, then there is a bijection $f : X \to X$. The inverse f^{-1} is also a bijection, so $\equiv$ is reflexive.

If $card(X) \equiv card(Y)$, then there is a bijection $f : X \to Y$. The inverse $f^{-1} : Y \to X$ is also a bijection, so $\equiv$ is symmetric.

If $card(X) \equiv card(Y)$ and $card(Y) \equiv card(Z)$, then there are bijections $f : X \to Y$ and $g : Y \to Z$. The functional composition $gf : X \to Z$ a bijection, so $\equiv$ is transitive.

2. *i*)Let *X* and *Y* be countable sets. Set $Y^* = Y - X$.

- If *X* and Y^* are both finite, $X \cup Y = X \cup Y^*$ is finite and countable.
- If one of *X* and Y^* are finite (without loss of generality assume *X* is finite, and that $X = \{x_0, \ldots, x_{n-1}\}$ and $Y^* = \{y_0, y_1, y_2, \ldots\}$), then

$$\begin{array}{cccccccc} 0 & 1 & 2 \ldots & n-1 & n & n+1 & n+2 & \ldots \\ \updownarrow & \updownarrow & \updownarrow \ldots & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\ x_0 & x_1 & x_2 & x_n & y_0 & y_1 & y_2 & \ldots \end{array}$$

demonstrates the countability of $X \cup Y = X \cup Y^*$.

- If *X* and Y^* are both infinite, then

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\ x_0 & y_0 & x_1 & y_1 & x_2 & y_2 & \ldots \end{array}$$

demonstrates the countability of $X \cup Y = X \cup Y^*$.

ii) We use the dovetailing construction of **Example 1.4.2** with *X* listed along the *x*-axis and *Y* listed along the *y*-axis. If either *X* or *Y* is finite, we remove the ordered pairs corresponding to the nonexistent elements from the enumeration.

iii) Let $X = \{x_0, \ldots, x_{n-1}\}$ or $X = \{x_0, x_1, \ldots\}$. We know that $\mathbb{N}$ is countably infinite, and we list it, expressing each number in binary in reverse order of the bits:

$$0 \leftrightarrow 0$$
$$1 \leftrightarrow 1$$
$$2 \leftrightarrow 01$$
$$3 \leftrightarrow 11$$
$$4 \leftrightarrow 001$$
$$5 \leftrightarrow 101$$

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We use each binary string $b_0b_1...b_k$ to represent the subset $\{x_i \mid x_i = 1\}$. That is, string 00101 represents subset $\{x_2, x_4\}$. So the finte subsets of X are

$$0 \leftrightarrow \varnothing$$
$$1 \leftrightarrow \{x_0\}$$
$$2 \leftrightarrow \{x_1\}$$
$$3 \leftrightarrow \{x_0, x_1\}$$
$$4 \leftrightarrow \{x_2\}$$
$$5 \leftrightarrow \{x_0, x_2\}$$

iv) Let $X = \{x_0, ..., x_{n-1}\}$ or $X = \{x_0, x_1, ...\}$, and let Y be the set of finite length sequences of elements of X. Since $\{\lambda, x_0, x_0x_0, x_0x_0x_0, ...\} \subseteq Y$, it follows that Y is infinite. The set of all sequences of length 0 are countable (there is only sequence λ), as well as the set of all sequences of length 1 (this is X, which is countable), as well as the set of all sequences of length 2 (using the dovetailing technique of **Example 1.4.2**), as well as the set of all sequences of length 3 (using the dovetailing technique of **Example 1.4.2** on sequences of lengths 2 and 1),… Let us denote these sequences as $\sigma_0, \sigma_1, \sigma_2, ...$ We finally use the construction of **Example 1.4.2** to dovetail these sequences together:

σ_0 λ

σ_1 x_0 x_1 ... x_{n-1}

σ_2 x_0x_0 $x_ox_1 ... x_0x_{n-1}$ x_1x_0 $x_1x_1 ... x_1x_{n-1} ... x_{n-1}x_0$ $x_{n-1}x_1 ... x_{n-1}x_{n-1}$

σ_3 ...

3. *i*) The components can be one-way infinite paths, two-way infinite paths and even length cycles. All the paths and cycles alternate between nodes of X and nodes of Y. Examples of these three types of components are:

- One-way infinite path

X is the even natural numbers and Y is the odd natural numbers and the arcs are $\{(n, n+1) \mid n \in \mathbb{N}\}$.

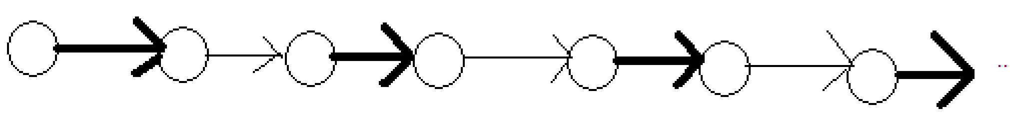
- Two-way infinite path

X is the even integers and Y is the odd integers and the arcs are $\{(n, n+1) \mid n \in \mathbb{Z}\}$.

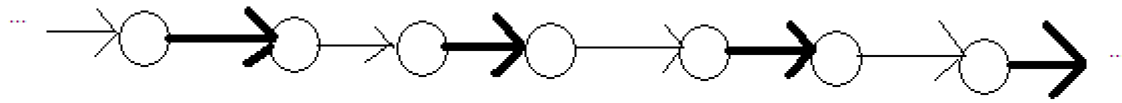
- Cycle

X=$\{0\}$ and Y=$\{1\}$ and the arcs are $\{[0,1],[1,0]\}$.

ii) If the component is a one-way infinite path, then the bijection contains the arc from the first node of the path and alternates arcs thereafter.



If the component is a two-way infinite path, the bijection contains the arcs from the nodes of X.



If the component is a cycle, the bijection contains the arcs from the nodes of X.

iii) $card(X) \le card(Y)$ and $card(Y) \le card(X)$ means that there exist total one-to-one functions from X into Y and from Y into X. These functions yield the arcs of the digraph. The union of the bijections from the different components yields a bijection between X and Y.

4. *i*) $\lambda \cup 0 \cup 1 \cup 00 \cup 01 \cup 10 \cup 11 \cup 000 \cup 001 \cup 010 \cup 011 \cup 100 \cup 101 \cup 110 \cup 111$

ii) $(0 \cup 1)^* 001 (0 \cup 1)^*$

iii) Every 1 must either be followed by a 0 or be the last character of the string.

$$(0 \cup 10)^* (\lambda \cup 1)$$

or, if you don't want to use λ in the regular expression, $(0 \cup 10)^* \cup (0 \cup 10)^* 1$.

iv) $(0 \cup 1)^* 1 (0 \cup 1)$

v) $\lambda \cup 0 \cup 1 (0 \cup 1)^* \cup 01 (0 \cup 1)^*$

5. u=00000000

6. $(A \cup \ldots \cup Z)(A \cup \ldots \cup Z)^* \leftarrow (1 \cup \ldots \cup 9)(0 \cup \ldots \cup 9)^* .(0 \cup \ldots \cup 9)^* (1 \cup \ldots \cup 9)$