CS3133

FINAL EXAM

Name______________

Date: October 13, 2011
All documentation permitted

1. (30 points) Prove or give a counterexample to each of the following.

<u>**CONJECTURE A**</u>: For any CFG $G = (N, \Sigma, P, S)$, if there exists a derivation

$S \overset{*}{\rightarrow} \alpha A \beta \overset{*}{\rightarrow} \alpha\gamma A \varphi\beta \overset{*}{\rightarrow} \alpha\gamma\phi\varphi\beta$ where $A \in N$, $\alpha, \gamma, \phi, \varphi, \beta \in \Sigma^*$, then $L(G)$ is infinite.

<u>**CONJECTURE B**</u>: For any CFG $G = (N, \Sigma, P, S)$, if there exists a derivation

$S \overset{*}{\rightarrow} \alpha A \beta \overset{*}{\rightarrow} \alpha\gamma A \varphi\beta \overset{*}{\rightarrow} \alpha\gamma\phi\varphi\beta$ where $A \in N$, $\alpha, \gamma, \phi, \varphi, \beta \in \Sigma^*$ and $|\gamma\varphi| \geq 1$
then $L(G)$ is infinite.

2. (30 points) Let L be the set of strings over $\{a, b, c\}$, with equal numbers of a's as b's as c's. So $bccaab \in L$ and $\varepsilon \in L$ and $abacacbbc \in L$, but $aabbccc \notin L$ and $abacbca \notin L$. Is L a context-free language? Justify your answer.

3. (40 points) We define a Turing Machine to be *conservative* if it always moves to the right. That is, a Turing Machine is conservative if its transition function is of the form $\delta : Q \times \Gamma \to Q \times \Gamma \times \{R\}$. Prove that a language is regular if and only if there is a total conservative Turing Machine to accept it.

<u>**CS3133**</u>

Solutions to <u>Final Exam</u>

1. CONJECTURE A is false. Consider the grammar $S \to S \mid a$. It admits the derivation $S \to S \to a$, which can be put in the form $S \xrightarrow{*} \alpha A\beta \xrightarrow{*} \alpha\gamma A\varphi\beta \xrightarrow{*} \alpha\gamma\phi\varphi\beta$ where A=S, $\alpha = \gamma = \varphi = \beta = \varepsilon \in \Sigma^*$ and $\phi = a$, but $L(G) = \{a\}$, which is finite.

CONJECTURE B is true. If $S \xrightarrow{*} \alpha A\beta \xrightarrow{*} \alpha\gamma A\varphi\beta \xrightarrow{*} \alpha\gamma\phi\varphi\beta$, then $A \xrightarrow{*} \gamma A\varphi \xrightarrow{*} \gamma\phi\varphi$, so we also have $A \xrightarrow{*} \gamma A\varphi \xrightarrow{*} \gamma\gamma A\varphi\varphi \xrightarrow{*} \gamma\gamma\phi\varphi\varphi$, and $A \xrightarrow{*} \gamma^k\phi\varphi^k$ for all $k \ge 0$. So, because $|\gamma\varphi| \ge 1$, the infinite set $\{\alpha\gamma^k\phi\varphi^k\beta \mid k \ge 0\} \subseteq L(G)$.

2. We show that the language is not context-free by invoking the Pumping Lemma for CFLs. If it were, then let n be the constant guaranteed by the Pumping Lemma. Choose $z = a^n b^n c^n$ in the language. Any way to decompose z into $uvwxy$ with $|vwx| \le n$ and $|vx| \ge 1$ has vwx spanning at most two different kinds of letters, and vx nonempty while spanning at most two different kinds of letters. The Pumping Lemma assures us that uv^2wx^2y belongs to the language, although it can't have the same number of a's as b's as c's, and hence uv^2wx^2y can't belong to the language. This contradiction yields that the language is not context-free.

3. The differences between a DFA and a conservative Turing Machine are that the latter can write on the tape (even though it can't ever read what it wrote) and that it can consume/read some blank symbols from the tape before entering the accept or reject state.

PROOF: Let M be the DFA $(Q, \Sigma, \delta, s, F)$. Then $L(M)$ is accepted by the total conservative Turing Machine $\left(Q \cup \{t, r\}, \Sigma, \Sigma \cup \{\vdash, \sqcup\}, \vdash, \sqcup, \Delta, s, t, r\right)$ with

- $\Delta(s, \vdash) = (s, \vdash, R)$,
- $\Delta(q, a) = (\delta(q, a), a, R)$ for all $q \in Q,\ a \in \Sigma$,
- $\Delta(q, \sqcup) = \begin{cases} (t, \sqcup, R), \text{if } q \in F \\ (r, \sqcup, R), \text{if } q \notin F \end{cases}$

Let M=$(Q, \Sigma, \Gamma, \vdash, \sqcup, \Delta, s, t, r)$ be a total conservative Turing Machine. Then $L(M)$ is accepted by the DFA $M = (Q, \Sigma, \delta, s, \{t\})$ where

- $\delta(t, a) = t$ for all $a \in \Sigma$,
- $\delta(r, a) = r$ for all $a \in \Sigma$,
- if $\Delta(q, a) = (p, a, R)$ for $a \in \Sigma$, $p \in Q$, $q \in Q - \{t, r\}$, then $\delta(q, a) = p$.