(18 points) (Kozen, problem 76) Let \( L = a^* b^* c^* - \{ a^i b^i c^i \mid i \geq 0 \} \). That is, \( L \) contains all strings of \( a \)'s followed by \( b \)'s followed by \( c \)'s which do not have exactly the same numbers of \( a \)'s and \( b \)'s and \( c \)'s. For example, \( aabbcL \neq \epsilon \) but \( aabb \in L \) and \( aabbbcc \in L \).

\( a \) Prove that \( L \) is a CFL by giving a CFG \( G \) such that \( L = L(G) \).

\( b \) Is your choice of \( G \) ambiguous?

\( c \) Prove that \( L \) is a CFL by giving a PDA \( M \) such that \( L = L(M) \).
1. **a** We use $A$ (respectively $B$ and $C$) to generate a string of $a$'s (respectively $b$'s and $c$'s). The trick will be to cover the three cases:

- $Z_1$ number of $a$'s is different than the number of $b$'s
  - $Y_{1a}$ more $a$'s than $b$'s
  - $Y_{1b}$ more $b$'s than $a$'s
- $Z_2$ number of $b$'s is different than the number of $c$'s
  - $Y_{2b}$ more $b$'s than $c$'s
  - $Y_{2c}$ more $c$'s than $b$'s
- $Z_3$ number of $a$'s is different than the number of $c$'s

\[
S \rightarrow Z_1 \mid Z_2 \mid Z_3 \\
Z_1 \rightarrow Y_{1a} C \mid Y_{1b} C \\
Y_{1a} \rightarrow a Y_{1a} b \mid a A \\
Y_{1b} \rightarrow a Y_{1b} b \mid b B \\
Z_2 \rightarrow A Y_{2b} \mid A Y_{2c} \\
Y_{2b} \rightarrow b Y_{2b} c \mid b B \\
Y_{2c} \rightarrow b Y_{2c} c \mid c C \\
Z_3 \rightarrow a Z_3 c \mid a A B \mid B c C \\
A \rightarrow a A \mid \varepsilon \\
B \rightarrow b B \mid \varepsilon \\
C \rightarrow c C \mid \varepsilon
\]

**b** The grammar above is ambiguous. There are three parse trees of $aab$:
M will first "guess" which of the six cases hold:

<table>
<thead>
<tr>
<th>Case</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>more a's than b's</td>
<td>$q_{a&gt;b}$</td>
</tr>
<tr>
<td>fewer a's than b's</td>
<td>$q_{b&gt;a}$</td>
</tr>
<tr>
<td>more b's than c's</td>
<td>$q_{b&gt;c}$</td>
</tr>
<tr>
<td>fewer b's than c's</td>
<td>$q_{c&gt;b}$</td>
</tr>
<tr>
<td>more a's than c's</td>
<td>$q_{a&gt;c}$</td>
</tr>
<tr>
<td>fewer a's than c's</td>
<td>$q_{c&gt;a}$</td>
</tr>
</tbody>
</table>

$$\delta(s, \varepsilon, \bot) = \{(q_{a>b}, \bot), (q_{b>a}, \bot), (q_{b>c}, \bot), (q_{c>b}, \bot), (q_{a>c}, \bot), (q_{c>a}, \bot)\}$$

We list the details of how to proceed from the first case, and the other five are defined in a similar way.

$q_{a>b}$ As a's are read, they are stacked.
$q_{a>b}$ As b's are read, a's are popped from the top of the stack.
$q_{a>b}$ A string of c's is read, and the stack is not touched.
If any a's are left in the stack, final state $q^*$ is entered.

$$\delta(q_{a>b}, a, \bot) = \{(q_{a>b}, a \bot)\}$$

$$\delta(q_{a>b}, a, a) = \{(q_{a>b}, aa)\}$$

$$\delta(q_{a>b}, b, a) = \{(q_{a>b}, a)\}$$

$$\delta(q_{a>b}, b, a) = \{(q_{a>b}, a)\}$$

$$\delta(q_{a>b}, c, a) = \{(q_{a>b}, c), (q^*, \varepsilon)\}$$

$$\delta(q_{a>b}, \varepsilon, a) = \{(q^*, \varepsilon)\}$$