CS3133

HW#5

DUE: Monday, October 6

1. (18 points) For each of the following languages over $\{0,1\}^*$, either give a PDA to accept it or prove that it is not context-free.

   a $\{0^n 1^n 0^n 1^n \mid n \geq 0\}$.

   b $\{0^m 1^n \mid 0 \leq m \leq n \leq 3m\}$.

   c The set of strings which have a prefix with more 1's than 0's. That is, the set of strings z which can be written $z = uv$ such that u has more 1's than 0's. For example, 100 and 0110 belong to our language, but 001 and 001101 do not.

2. (8 points) Prove or give a counterexample to the following.

CONJECTURE: For any context-free language L_0 and regular language L_1, the language $L_0 \cup L_1$ is context-free.

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Solutions for HW#5

1. ***a*** $\{0^n1^n0^n1^n \mid n \geq 0\}$ is not context free. Assume that L is context free, and let k be as guaranteed by the Pumping Lemma for context free languages. We know that $0^k1^k0^k1^k \in L$. So the P.L. assures us that it can be expressed as $0^k1^k0^k1^k = uvwxy$ with $|vwx| \leq k$ and $|vx| \geq 1$. Thus, vx is not empty. If either v or x contains 0s **and** 1s, then uv^2wx^2y is not of the correct form (0s followed by 1s followed by 0s followed by 1s). Hence, at least one of the four sequences of 0s or 1s is not spanned by v **or** by x, and uv^2wx^2y can not have the same number of 0s or 1s in each of the four sequences. This is a contradiction, and $\{0^n1^n0^n1^n \mid n \geq 0\}$ can not be context free.

b $\{0^m1^n \mid 0 \leq m \leq n \leq 3m\}$ is context-free. The idea behind a PDA to accept it is that for every 0 which is read, we use nondeterminism to push one, two or three new 0s (while in state s) onto the stack. We then pop a 0 for each 1 which is read (while in state q). $\{0^m1^n \mid 0 \leq m \leq n \leq 3m\}$ is accepted by final state by

$M = (\{s,q,r\},\{0,1\},\{\bot,0\},\delta,s,\bot,\{r\})$ with

$$\delta(s,0,\bot) = \{(s,0\bot),(s,00\bot),(s,000\bot)\}$$
$$\delta(s,0,0) = \{(s,00),(s,000),(s,0000)\}$$
$$\delta(s,1,0) = \{(q,\varepsilon)\}$$
$$\delta(q,1,0) = \{(q,\varepsilon)\}$$
$$\delta(q,\varepsilon,\bot) = \{(r,\varepsilon)\}.$$

c The language is accepted by empty stack by the deterministic PDA

$$M = (\{s,f\},\{0,1\},\{0,\bot\},\delta,s,\bot,\{f\})$$

where $\delta(s,0,\bot) = \{(s,0\bot)\}$, $\delta(s,0,0) = \{(s,00)\}$ — push 0's onto the stack

$\delta(s,1,0) = \{(s,\varepsilon)\}$ — pop a 0 for each 1

$\delta(s,1,\bot) = \{(f,\bot)\}$ — more 1's than 0's, go to f

$\delta(f,0,\bot) = \delta(f,1,\bot) = \{(f,\bot)\}$ — consume the rest of input

2. The CONJECTURE is true. If L_1 is regular, then it's context-free. It's accepted by a PDA which accepts by final state and never uses the stack; for each transition $\delta(q,a) = r$ of the DFA which accepts L_1 we add the transition $\delta(q,a,\bot) = \{(r,\bot)\}$ to the PDA. So there are context-free grammars $G_0 = (N_0,\Sigma,P_0,S_0)$ and $G_1 = (N_1,\Sigma,P_1,S_1)$ such that $L_0 = L(G_0)$ and $L_1 = L(G_1)$. We assume without loss of generality that $N_0 \cap N_1 = \varnothing$. The grammar

$$(N_0 \cup N_1, \Sigma, P_0 \cup P_1 \cup \{S \to S_0, S \to S_1\}, S)$$

generates $L_0 \cup L_1$ if $S \notin N_0 \cup N_1$.