1. (3 points) Assume that $L$ is a regular language. Show that the set of prefixes of strings in $L$ is also regular. That is, $\{u \mid \exists v (uv \in L)\}$ is regular.

2. (3 points) Show that the set of all strings $z$ over \{0,1\} such that $|z| \geq 3$ and the third character from the end of $z$ is a 0 is a regular language. For example, 001100 does not belong to the language, but 001000 does.

3. (3 points) Give a regular expression for the language $L = \{0,1\}^*$ such that $L$ contains all strings with an even number of 0's and every 0 is followed by at least one 1. For example, $\{\varepsilon, 011101, 1, 010110101\} \subseteq L$ but 010 $\not\in L$.

4. (9 points) Prove that for any regular language $L$, there is an NFA $M = (Q, \Sigma, \delta, s, F)$ such that $L = L(M)$ and $|F| = 1$. Don't just give a construction; prove that $L$ belongs to the language described by or accepted by your construction.
CS3133
Solutions for HW#2

1. If $L$ is regular, then it is accepted by a DFA $M = (Q, \Sigma, \delta, s, F)$. Also, $u$ is a prefix of $uv \in L$ if and only if there is a path in $M$ from $s$ to $\tilde{\delta}(s, u)$ and there is a path in $M$ from $\tilde{\delta}(s, u)$ to a state in $F$. Let $S \subseteq Q$ be the states such that for each $q \in S$, there is a path in $M$ from $s$ to $q$ and there is a path in $M$ from $q$ to a state of $F$. Language $\{u \mid \exists v (uv \in L)\}$ is accepted by DFA $(Q, \Sigma, \delta, s, F \cup S)$, and hence it is regular.

2. The idea is to design an NFA $M$ which guesses that a symbol being read is the third from the end, and then verifies that it's a 0 (go to $q_1$) and that exactly two more symbols follow. $M = (\{s, q_1, q_2, q_3\}, \{0, 1\}, \delta, \{q_1\})$, where

\[
\begin{array}{c|c|c}
\delta & 0 & 1 \\
\hline
s & \{s, q_1\} & \{s\} \\
q_1 & \{q_2\} & \{q_2\} \\
q_2 & \{q_3\} & \{q_3\} \\
q_3 & \emptyset & \emptyset \\
\end{array}
\]

3. $1^*(011^*011^*)^*$

4. If $L$ is regular, then there is an NFA $M^* = (Q, \Sigma, \Delta^*, S, F)$ without $\varepsilon$-transitions such that $L = L(M^*)$. We construct $M = (Q \cup \{f\}, \Sigma, \Delta, S^*, \{f\})$ where

$S^* = \begin{cases} S, & \text{if } S \cap F = \emptyset \\ S \cup \{f\}, & \text{otherwise} \end{cases}$.

For any transition to a final state of $M^*$, we add to $M$ an "equivalent" transition to the new, unique final state $f$. Also, $M$ doesn't permit any transitions from $f$. That is, for all $A \subseteq Q \cup \{f\}, a \in \Sigma$,

$\Delta(A, a) = \begin{cases} \Delta^*(A, a), & \text{if } \Delta^*(A, a) \cap F = \emptyset \\ \Delta^*(A, a) \cup \{f\}, & \text{otherwise} \end{cases}$

and $\Delta(\{f\}, a) = \emptyset$.

We use induction to show that for any $w \in \Sigma^*$,

$\hat{\Delta}(S, w) = \begin{cases} \hat{\Delta}^*(S, w), & \text{if } \hat{\Delta}^*(S, w) \cap F = \emptyset \\ \hat{\Delta}^*(S, w) \cup \{f\}, & \text{otherwise} \end{cases}$.
That is, if we ignore \( f \), then \( \hat{\Delta} \) and \( \hat{\Delta}^* \) do exactly the same thing. As a basis, \( w = \varepsilon \), we first note that from the definition of \( S^* \),

\[
\hat{\Delta}(S, \varepsilon) = \begin{cases} 
S, & \text{if } S \cap F = \emptyset \\
S \cup \{ f \}, & \text{otherwise}
\end{cases}
\]

For an induction hypothesis, we assume that

\[
\hat{\Delta}(S, w) = \begin{cases} 
\hat{\Delta}^*(S, w), & \text{if } \hat{\Delta}^*(S, w) \cap F = \emptyset \\
\hat{\Delta}^*(S, w) \cup \{ f \}, & \text{otherwise}
\end{cases}
\]

For any \( a \in \Sigma \), even if \( f \in \hat{\Delta}(S, w) \), there are no transitions out of \( f \), \( \Delta(\{ f \}, a) = \emptyset \). Hence, \( \hat{\Delta}(S, wa) = \Delta(\hat{\Delta}(S, w), a) \) is equal to \( \hat{\Delta}^*(S, w) \) if \( \hat{\Delta}^*(S, w) \cap F = \emptyset \) and is equal to \( \hat{\Delta}^*(S, w) \cup \{ f \} \) if \( \hat{\Delta}^*(S, w) \cap F \neq \emptyset \), which is what we are trying to prove.