CS3133
HW#1 SOLUTIONS

1. ***a***) If X and Y are countably infinite, then they can be enumerated by the bijections

$$\begin{array}{llll} X & x_0 & x_1 & x_2\ldots.. \\ f & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ N & 0 & 1 & 2\ldots\ldots.. \end{array}$$

and

$$\begin{array}{llll} Y & y_0 & y_1 & y_2\ldots.. \\ g & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ N & 0 & 1 & 2\ldots\ldots.. \end{array}$$

Since X and Y are disjoint, their union can be enumerated by the bijection

$$\begin{array}{lllll} X \cup Y & x_0 & y_0 & x_1 & y_1\ldots.. \\ h & \mathbf{b} & \mathbf{b} & \mathbf{b} & \mathbf{b} \\ N & 0 & 1 & 2 & 3\ldots\ldots.. \end{array}$$

where element $z_i \in X \cup Y$ corresponds to $x_{\lfloor i/2 \rfloor}$ if i is even and $y_{\lfloor i/2 \rfloor}$ if i is odd.

b) If X and Y are infinite, then the result follows from part ***a***). If X and Y are finite, then $X = \{x_1,\ldots,x_m\}$ and $Y = \{y_1,\ldots,y_n\}$, and their union can be enumerated by

$$\begin{array}{llllllll} X \cup Y & x_1 & x_2\ldots. & x_m & y_1 & y_2 & \ldots.. & y_n \\ h & \mathbf{b} & \mathbf{b}.. & \mathbf{b} & \mathbf{b} & \mathbf{b}\ldots.. & & \mathbf{b} \\ N & 0 & 1\ldots..m-1 & m & m+1\ldots..m+n-1 & & & \end{array}$$

If exactly one of X and Y are finite (without loss of generality, assume that $X = \{x_1,\ldots,x_m\}$ and Y is countably infinite), then their union can be enumerated by

$$\begin{array}{llllll} X \cup Y & x_1 & x_2\ldots. & x_m & y_0 & y_1 \;\ldots.. \\ h & \mathbf{b} & \mathbf{b}.. & \mathbf{b} & \mathbf{b} & \mathbf{b}\ldots.. \\ N & 0 & 1\ldots..m-1 & m & m+1\ldots.. & \end{array}$$

2. $\{f : N \to N | f \text{ is monotone increasing and total}\}$ is not countable. If it were countable, then we could order its members $f_0, f_1, f_2,\ldots$ But consider the total function

$$g^*(n) = \begin{cases} f_0(0)+1, \text{ if } n = 0 \\ g^*(n-1) + f_n(n)+1, \text{ if } n > 0 \end{cases}$$

g^* is certainly total and since $(\forall n > 0)(g^*(n) > g^*(n-1))$, it is monotone increasing. But for all $n \in N$, $g^*(n) > f_n(n)$, so g^* can not belong to the order $f_0, f_1, f_2,\ldots$ Hence, $\{f : N \to N | f \text{ is monotone increasing and total}\}$ can not be countable.