1. (12 points) Given an array \( T \) of \( n \) numbers, and given integer \( k \), \( n \geq k \geq 1 \), we seek the \( k^{th} \) smallest member of \( T \). Consider the following two algorithms:

**ALGORITHM A:**

```
HEAPSORT T
return T[k]
```

**ALGORITHM B:**

```
INSERTIONSORT T
return T[k]
```

Answer true or false to each of the following three statements:

(a) ALGORITHM A will solve the problem in worst-case time in \( O(n + k \log n) \).

(b) ALGORITHM B will solve the problem in worst-case time in \( O(n + k \log n) \).

(c) ALGORITHM B will solve the problem in best-case time in \( O(n + k \log n) \).
2. (25 points) Noting that the minimum spanning trees of the following two graphs are equal,

Ben makes the following claim:

**Conjecture:** For any graph $G=<N,A>$ with function $\text{length}: A \rightarrow R^+$, let $a \in A$ be an arc of maximum length (every other arc in $A$ is shorter than $a$). Let $H$ be formed from $G$ by removing $a$. The minimum spanning tree of $G$ must be equal to the minimum spanning tree of $H$. That is, we can remove the longest arc in any graph without changing its minimum spanning tree.

Prove or give a counterexample to Ben’s Conjecture.
3. (28 points) Suppose that you want to implement the abstract data type Priority Queue using an ordered linked list as a data structure, with standard linked list operations. That is, after executing:
   - construct($Q$)
   - insert(18,$Q$)
   - insert(1,$Q$)
   - insert(28,$Q$)
   - insert(9,$Q$)

   The data structure holding $Q$ would look like:

   ![Linked List Diagram]

   Using $\Theta$–notation, what is the worst-case time to implement the following instructions? (For “insert” and “delete_min”, assume that $Q$ contains $n$ elements.)
   - construct($Q$)
   - insert($x$,$Q$)
   - delete_min($Q$)
4. (35 points) State whether each of the following claims is true or false, and justify your response.

(a) \( \frac{n^2 + n}{2} \in O(6n) \)

(b) \( 6n \in O\left( \frac{n^2 + n}{2} \right) \)
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Solutions to Midterm Exam

1. (a) false For \( k=1 \), the time constraint is \( O(n) \), and HEAPSORT takes worst-case time in \( O(n \lg n) \)

(b) false For \( k=n \), the time constraint is \( O(n \lg n) \), and INSERTIONSORT takes worst-case time in \( O(n^2) \).

(c) true The best-case execution time of INSERTIONSORT is \( O(n) \), and for any \( k \) between 1 and \( n \), the time constraint is at least linear. That is, \( n \in O(n + k \lg n) \).

2. The conjecture is false, although it is true for all graphs in which a longest arc belongs to a cycle. To see that it is false, we note that removing the longest arc from \( G \) disconnects the graph (\( H \) would not have a spanning tree).

3. 
   \[
   \begin{align*}
   \text{construct}(Q) & : \Theta(1) \\
   \text{insert}(x, Q) & : \Theta(n) \\
   \text{delete}_{\text{min}}(Q) & : \Theta(1)
   \end{align*}
   \]

4. (a) false If it were true, there would exist \( c, n_0 \) such that \( \frac{n^2 + n}{2} \leq c6n \) for all \( n \geq n_0 \).

Dividing both sides of the inequality by \( 6n \), this would imply that for all \( n \) sufficiently large, \( \frac{n}{12} \leq \frac{n}{12} + \frac{1}{12} \leq c \). But this is impossible. For any \( c \), simply choose \( n > 12c \) to derive a contradiction.

(b) true We must show that there exist \( c, n_0 \) such that \( 6n \leq c \frac{n^2 + n}{2} \) for all \( n \geq n_0 \).

Dividing both sides of the inequality by \( 6n \),

\[
1 \leq c \frac{n}{12} + \frac{c}{12}
\]

Choosing \( c=12 \) and \( n_0=1 \) yields the desired result.