CS2223

MIDTERM EXAM

Name______________

Date: April 6, 2006
All documentation permitted

1	(25)	________
2	(25)	________
3	(30)	________
4	(20)	________

TOTAL ________

1. (25 points) Suppose you are given two lists $A[1..n]$ and $B[1..n]$ of $n>1$ integers. The output is an array $C[1..2n]$. The entries $C[1..m]$ should contain the distinct members of $\{A[1],...,A[n],B[1],...,B[n]\}$. Show that a lower bound on the complexity of solving this problem is in $\Omega(n)$.

2. (25 points) Show that the complexity of finding the $42^{\underline{nd}}$ smallest element of an array $A[1..n]$, $n \geq 42$, is in $O(n)$.

3. (30 points) Prove or give a counterexample to each of the following.

CONJECTURE ***a***: For all functions $f, g : \mathbb{Z}^+ \to \mathbb{Z}^+$, if $f \in O(g)$ then $O(f) \subseteq O(g)$.

CONJECTURE ***b***: For all functions $f, g : \mathbb{Z}^+ \to \mathbb{Z}^+$, if $f \in O(g)$ then $O(g) \subseteq O(f)$.

4. (20 points) Suppose you want to implement an abstract data type to support the following operations

- CONSTRUCT(H)
- INSERT(x,H)
- DELETE-MAX(H)
- DELETE-MIN(H)

Alex proposes that you can support all the operations using a max-heap, with the INSERT(x,H) and DELETE-MAX(H) taking time in $O(\lg n)$, and CONSTRUCT(H) and DELETE-MIN(H) taking time in $O(1)$. The justification for the last claim is that if there are n elements in a max-heap H with n elements, then the smallest element is in $H[n]$. Either prove that Alex is correct or provide a counter-example proving him wrong.

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Solutions to Midterm Exam

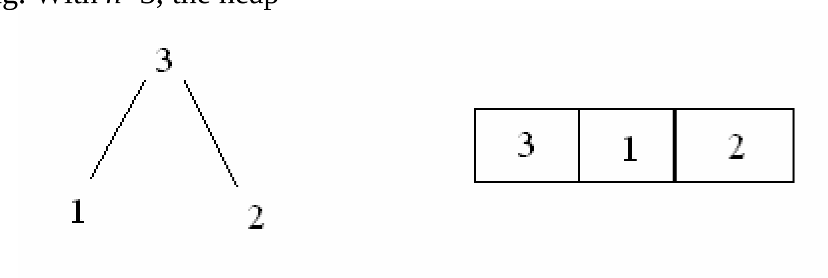
1. Assume there exists an algorithm to solve the problem in time which doesn't belong to $\Omega(n)$. Then some element of A or of B wasn't examined. Without loss of generality, assume that $A[i^*]$ wasn't examined. If we change $A[i^*]$ to be some element which is neither in A nor in B but keep every other element of A and of B the same, then the purported algorithm returns the same answer as before. But since the answer is now wrong, the contradiction implies that the algorithm can't exist.

2.	BUILD-MIN-HEAP(A)	$O(n)$
	for $i \leftarrow 1$ **to** 41 **do** DELETE-MIN(A)	$O(\lg n)$
	return DELETE-MIN(A)	$O(\lg n)$

3. ***a*** The CONJECTURE is true. If $f \in O(g)$ then there exist c_0 and n_0 such that for all $n \geq n_0$, $f(n) \leq c_0 g(n)$. For any $h \in O(f)$, there exist c_1 and n_1 such that for all $n \geq n_1$, $h(n) \leq c_1 f(n)$. So for all $n \geq \max(n_0, n_1)$, $h(n) \leq c_0 c_1 g(n)$ and $h \in O(g)$. So $O(f) \subseteq O(g)$.

b The CONJECTURE is false. Letting $f(n) = n$ and $g(n) = n^2$, it follows that $n^2 \in O(n^2)$ but $n^2 \notin O(n)$. Thus $O(n^2)$ is not a subset of $O(n)$.

4. Alex is wrong. With n=3, the heap



has the minimum element in H[2], not H[3].