CS2223
MIDTERM EXAM

Name________________

Date: April 6, 2006
All documentation permitted

1     (25)     ________
2     (25)     ________
3     (30)     ________
4     (20)     ________

TOTAL     ___________
1. (25 points) Suppose you are given two lists $A[1..n]$ and $B[1..n]$ of $n > 1$ integers. The output is an array $C[1..2n]$. The entries $C[1..m]$ should contain the distinct members of \{ $A[1], ..., A[n], B[1], ..., B[n]$ \}. Show that a lower bound on the complexity of solving this problem is in $\Omega(n)$. 
2. (25 points) Show that the complexity of finding the 42nd smallest element of an array $A[1..n]$, $n \geq 42$, is in $O(n)$. 
3. (30 points) Prove or give a counterexample to each of the following.

**CONJECTURE a:** For all functions \( f, g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \), if \( f \in O(g) \) then \( O(f) \subseteq O(g) \).

**CONJECTURE b:** For all functions \( f, g : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \), if \( f \in O(g) \) then \( O(g) \subseteq O(f) \).
4. (20 points) Suppose you want to implement an abstract data type to support the following operations

- `CONSTRUCT(H)`
- `INSERT(x,H)`
- `DELETE-MAX(H)`
- `DELETE-MIN(H)`

Alex proposes that you can support all the operations using a max-heap, with the `INSERT(x,H)` and `DELETE-MAX(H)` taking time in $O(\lg n)$, and `CONSTRUCT(H)` and `DELETE-MIN(H)` taking time in $O(1)$. The justification for the last claim is that if there are $n$ elements in a max-heap $H$ with $n$ elements, then the smallest element is in $H[n]$. Either prove that Alex is correct or provide a counter-example proving him wrong.
1. Assume there exists an algorithm to solve the problem in time which doesn't belong to $\Omega(n)$. Then some element of $A$ or of $B$ wasn't examined. Without loss of generality, assume that $A[i^*]$ wasn't examined. If we change $A[i^*]$ to be some element which is neither in $A$ nor in $B$ but keep every other element of $A$ and of $B$ the same, then the purported algorithm returns the same answer as before. But since the answer is now wrong, the contradiction implies that the algorithm can't exist.

2. BUILD-MIN-HEAP($A$) \hspace{2cm} O(n)
   \begin{align*}
   \text{for } i & \leftarrow 1 \text{ to 41 do DELETE-MIN}(A) \hspace{1cm} O(\lg n) \\
   \text{return DELETE-MIN}(A) \hspace{1cm} O(\lg n)
   \end{align*}

3. \textbf{a} The CONJECTURE is true. If $f \in O(g)$ then there exist $c_0$ and $n_0$ such that for all $n \geq n_0$, $f(n) \leq c_0 g(n)$. For any $h \in O(f)$, there exist $c_1$ and $n_1$ such that for all $n \geq n_1$, $h(n) \leq c_1 f(n)$. So for all $n \geq \max(n_0, n_1)$, $h(n) \leq c_0 c_1 g(n)$ and $h \in O(g)$. So $O(f) \subseteq O(g)$.

   \textbf{b} The CONJECTURE is false. Letting $f(n) = n$ and $g(n) = n^2$, it follows that $n^2 \in O(n^2)$ but $n^2 \not\in O(n)$. Thus $O(n^2)$ is not a subset of $O(n)$.

4. Alex is wrong. With $n=3$, the heap

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  3
 / \ /
1  2 3
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