CS2223
HW#4

DUE: Tuesday, November 27

1 (4 points) Suppose the following algorithm is given a complete graph $G = (N, A)$ with distinct edge lengths, that is $(\forall a_1, a_2 \in A)\, length(a_1) \neq length(a_2)$. Prove that it either always produces a minimum spanning tree T or provide a counterexample to show that it fails.

$T \leftarrow \varnothing$
$N^* \leftarrow$ an arbitrary vertex of N
for $i \leftarrow 1$ **to** n-1 **do**
  let $a = \{v_i, v_j\}$ be a shortest edge from v_i, the last vertex to be added
    to N^*, and some vertex $v_j \in N \setminus N^*$
  $T \leftarrow T \cup \{a\}$
  $N^* \leftarrow N^* \cup \{v_j\}$

That is, the algorithm always adds a shortest edge between the vertex most recently added to N^* and a vertex not yet in N^*.

2. (5 points) Prove or give a counterexample to the following.

CONJECTURE: For any connected graph $G = (N, A)$ with distinct edge lengths, if there is an edge $a \in A$ and a cycle in G such that a is the shortest edge of the cycle, then a belongs to every minimum spanning tree.

3. (6 points) Suppose that in a *vertex-weighted graph* $G = (N, A)$ we extend the definition of *length* to be $length : N \cup A \to \mathbb{R}^+$, that is, we associate a length with each vertex **and** each edge. We define the length of path $v_i, a_i, v_{i+1}, a_{i+1}, \ldots, a_{j-1}, v_j$ to be $\sum_{i \le k \le j} v_k + \sum_{i \le k < j} a_k$.

That is, the length of a path is the sum of the lengths of the vertices on the path **plus** the sum of the lengths of the edges on the path. Describe an algorithm to solve the single-source shortest path problem in a vertex-weighted graph. The worst-case execution time of your algorithm should be in $\Theta(n^2)$.

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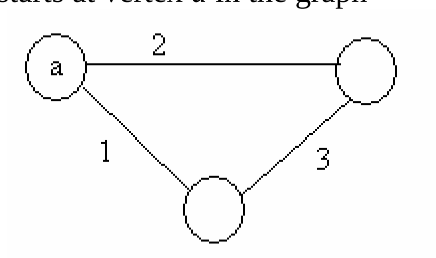
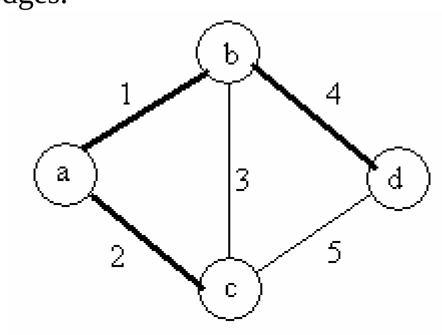
HW#4 SOLUTIONS

1. The algorithm fails if it starts at vertex *a* in the graph



In fact, the algorithm always produces a path, and **only** succeeds when a minimum spanning tree is a path starting at the initial vertex.

2. The CONJECTURE is false. The edge $\{b, c\}$ is a shortest edge of the cycle through vertices b, c, d of the graph below, but it does not belong to the minimum spanning tree, which is shown with bold edges.



3. We first convert the graph into a directed graph, and then we "absorb" the length of each vertex into the length of the edges leaving the vertex. Finally we solve the problem using DIJKSTRA'S ALGORITHM, where we augment each $D[v]$ by *length*(v).

for each undirected $a = \{v, w\} \in A$

replace a in A by the pair of directed edges (v,w) and (w,v)

$length((v,w)) = length(\{v,w\}) + length(v)$

$length((w,v)) = length(\{v,w\}) + length(w)$

DIJKSTRA'S ALGORITHM

for each $v \in n$

$D[v] = D[v] + length(v)$