CS2223
HW#4

DUE: Monday, December 1

1. (10 points) Suppose you are given a graph $G = (V, E)$ with a weight function $w : V \cup E \to \mathbb{R}^+$ which associates a weight with each edge **and** each vertex. The weight of a path from x_1 to x_m consisting of edges $(x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m)$ is $\sum_{1 \le i < m} w(x_i, x_{i+1}) + \sum_{1 < i < m} w(x_i)$. That is, it is the weights of all the edges of the path plus the weights of all the vertices the path passes through en route from x_1 to x_m. This corresponds to problems in which a cost is incurred for traversing an edge and for traversing intermediate vertices. Describe an efficient algorithm for finding the lightest (shortest) path from $\sigma \in V$ to every other vertex in such a graph.

2. (3 points) Prove or give a counterexample to the following:

   CONJECTURE: In any directed graph $G = (V, E)$ with $w : E \to \mathbb{R}$, if G does not admit a cycle of negative length, then Dijkstra's Algorithm returns the length of the shortest path from source σ to every $v \in V$.

Note that the new problem permits negative edge lengths. Only negative length cycles are prohibited.

3. (10 points) Do **Problem *24-3*** on pg. 615 of our text. (Hint: How do you convert multiplication to addition?)

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HW#4 SOLUTIONS

1. We replace each vertex $v \in V$ with two new vertices v_{in}, v_{out}, and each edge $(u,v) \in E$ with two new edges $(u_{out}, v_{in}), (v_{out}, u_{in})$. The weights of the edges are $w^*(u_{out}, v_{in}) = w(u,v)$, and we use Dijkstra's algorithm from source σ_{out} to every v_{in}. For each $v \in V$, there is an edge (v_{in}, v_{out}) of weight $w^*(v_{in}, v_{out}) = w(v)$. In this new graph with weights w^* on the edges, a shortest path from σ_{out} to v_{in} corresponds to a shortest path in the original graph (with weights w on the vertices and edges).

2. The conjecture is false. For graph $G = (V, E)$ with $V = \{\sigma, a, \tau\}$ and edges $E = \{(\sigma,\tau), (\sigma, a), (a, \tau)\}$ with weights

	(σ,τ)	(σ, a)	(a,τ)
w	5	8	-6

, we note that G does not admit any cycles of negative length, but it returns

	σ	a	τ
D	0	8	5

3. We form a complete directed graph $G = (V, E)$ with a vertex c_i corresponding to each currency. To associate weights with the edges, we note that

$$R[i_1, i_2] * R[i_2, i_3] * \ldots * R[i_{k-1}, i_k] * R[i_k, i_1] > 1$$

if and only if

$$\frac{1}{R[i_1, i_2]} * \frac{1}{R[i_2, i_3]} * \ldots * \frac{1}{R[i_{k-1}, i_k]} * \frac{1}{R[i_k, i_1]} < 1$$

which, taking logs of both sides, is true if and only if

$$\lg \frac{1}{R[i_1, i_2]} + \lg \frac{1}{R[i_2, i_3]} + \ldots + \lg \frac{1}{R[i_{k-1}, i_k]} + \lg \frac{1}{R[i_k, i_1]} < 0.$$

We define the weights of the edges to be $w(c_i, c_j) = -\lg R[i, j]$, and the question reduces to whether or not the graph admits a negative length cycle.