CS2223
HW#2

DUE: Friday, March 24

1. (8 points) For each of the following assertions, prove that it is true or false.

a) $\sum_{3 \le k \le n} k \in O(n^3)$

b) $\sum_{3 \le k \le n} k \in \Theta(n^3)$

c) $\sum_{3 \le k \le n} k \in \Theta(n^2)$

d) $\sum_{3 \le k \le n} k \in \Omega(n)$

2. (5 points) Prove or give a counterexample to the following

   CONJECTURE: For any $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and any c>0, $\Theta(f(n)) = \Theta(c * f(n))$.

3. (5 points) Describe an algorithm (you don't have to program it) to find the median of an array *A* of 2**m*+1 numbers which uses 2**m* pairwise comparisons in the best case. The median is an element *A*[*i*] such that *m* elements of *A* are less than or equal to *A*[*i*], and *m* elements of *A* are greater than or equal to *A*[*i*].

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HW#2 SOLUTIONS

1. From our discussion of arithmetic series, $\sum_{3\le k\le n} k = \sum_{0\le k\le n} k - \sum_{0\le k\le 2} k = \frac{n(n+1)}{2} - 3$.

a The assertion is true. Choosing c=1 and $n_0 = 1$. For all $n \ge n_0$,

$$\frac{n(n+1)}{2} - 3 \le \frac{n^2}{2} + \frac{n}{2} \le \frac{n^2}{2} + \frac{n^2}{2} = n^2 \le n^3$$

b The assertion is false. If it were true, then there would exist c and n_0 such that for all $n \ge n_0$, $n^3 \le c\left(\frac{n(n+1)}{2} - 3\right) = \frac{c}{2}n^2 + \frac{c}{2}n - 3c$. Dividing both sides of the preceding inequality by n^2 yields $n \le \frac{c}{2} + \frac{c}{2}\frac{1}{n} - 3c\frac{1}{n^2}$. For n sufficiently large, the last two terms on the right side of the inequality go to 0. For all $n > \frac{c}{2}$, the inequality can not hold, and by contradiction the assertion is false.

c The assertion is true. Choosing , c_0=1/2, c_1=1 and $n_0 = 3$,

$$c_0 n^2 = \frac{1}{2}n^2 \le \sum_{3\le k\le n} k = \frac{1}{2}n^2 + \frac{1}{2}n - 3 \le \frac{1}{2}n^2 + \frac{1}{2}n \le n^2 = c_1 n^2.$$

d The assertion is true. Choosing , c=1/2 and $n_0 = 3$,

$$cn = \frac{1}{2}n \le \frac{1}{2}n + \left(\frac{1}{2}n^2 - 3\right) = \sum_{3\le k\le n} k \ .$$

2. The CONJECTURE is true.

$$t(n) \in \Theta(f(n)) \Leftrightarrow (\exists c_0)(\exists c_1)(\exists n_0)(\forall n \ge n_0) c_0 f(n) \le t(n) \le c_1 f(n)$$

and

$$t(n) \in \Theta(cf(n)) \Leftrightarrow (\exists \kappa_0)(\exists \kappa_1)(\exists n_0)(\forall n \ge n_0) \kappa_0 cf(n) \le t(n) \le \kappa_1 cf(n)$$

Clearly, given κ_0, κ_1 and n_0 we can choose $c_0 = \kappa_0 c$ and $c_1 = \kappa_1 c$ so that $t(n) \in \Theta(cf(n)) \Rightarrow t(n) \in \Theta(f(n))$. Likewise, given c_0, c_1 and n_0 we can choose $\kappa_0 = c_0 / c$ and $\kappa_1 = c_1 / c$ so that $t(n) \in \Theta(f(n)) \Rightarrow t(n) \in \Theta(cf(n))$.

3. One algorithm "guesses" that $A[1]$ is the median and tests that exactly m elements of A are less than $A[1]$, using $2*m$ comparisons. If the number of elements of $A[1]$ less than $A[1]$ is not m, then it INSERTIONSORTS A and returns $A[m+1]$.

```
LessThanA1 ← 0
for i ← 2 to 2*m+1 do
    if A[i] < A[1] then LessThanA1 ← LessThanA1+1
 if LessThanA1=m then return A[1]
                 else {INSERTIONSORT(A)
                       return A[m+1]}
```