CS2022/MA2201

Midterm Exam

Name______________

Date: April 5, 2007

All documentation permitted

1. (20 points) Let $A = \{n \in \mathbb{Z} \mid n \text{ is even}\}$ and $B = \{n \in \mathbb{Z}^+ \mid n \geq 100\}$. Is $A \cap B$ finite, countably infinite or uncountably infinite? Justify your answer.

2. (15 points) State whether each of the following propositions is true or false, where the domain of discourse is the integers, $\mathbb{Z}$.

a $\exists n \left(\frac{1}{n^2+1} \right) > 1$

b $\forall n \left(n^2 \geq n \right)$

c $\forall a \ \forall b \ \forall c \left(\left(a = b \right) \rightarrow \left(a * c = b * c \right) \right)$

3. (25 points) ***a*** Is $\sum_{3 \le k \le n} 2k$ a one-to-one function from $n \in \{3, 4, 5, ...\}$ to the integers? Justify your answer.

b Is $\sum_{3 \le k \le n} 2k$ an onto function from $\{3, 4, 5, ...\}$ to $\mathbb{Z}^+$? Justify your answer.

c Is $\sum_{3 \le k \le n} 2k$ a bijection from $\{3, 4, 5, ...\}$ to $\mathbb{Z}^+$? Justify your answer.

d What is a closed form for $\sum_{3 \le k \le n} 2k$ defined on domain $\{3, 4, 5, ...\}$?

4. (10 points) Decide whether the following argument is valid, and justify your response.

If you do well in Discrete, you'll be admitted to grad school. If you work hard in Discrete, you'll do well. Al works hard in Discrete. Therefore Al will be admitted to grad school.

5. (10 points) Decide whether the following statement is a tautology, and justify your reply.

$$\forall x\left(\left(P(x) \rightarrow Q(x)\right) \leftrightarrow \left(Q(x) \rightarrow P(x)\right)\right)$$

6. (10 points) Prove or give a counterexample to the following.

<u>CONJECTURE</u>: For all sets *A*, *B* and *C*, if $A \cup C = B \cup C$, then *A*=*B*.

7. (10 points) ***a*** Give a function $f:\mathbb{N}\rightarrow\mathbb{N}$ which is one-to-one but not onto. Note: $\mathbb{N}=\{0,1,2,...\}$

b Give a function $f:\mathbb{N}\rightarrow\mathbb{N}$ which is onto but not one-to-one.

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Solutions to Midterm Exam

1. $A \cap B = \{100, 102, 104, 106, ...\}$ is countably infinite. A bijection between $A \cap B$ and $\mathbb{Z}^+$ is

$A \cap B$	100	102	104	106	108	110	112	...
$\mathbb{Z}^+$	1	2	3	4	5	6	7	...

2. ***a*** false ***b*** true ***c*** true

3. ***a*** Yes If $\sum_{3 \le k \le n} 2k$ were not one-to-one, there would be $m, n \in \{3, 4, 5, ...\}$, $m \ne n$, such that $\sum_{3 \le k \le m} 2k = \sum_{3 \le k \le n} 2k$. Without loss of generality, assume that $m>n$. Then $\sum_{3 \le k \le m} 2k = \sum_{3 \le k \le n} 2k + \sum_{n \le k \le m} 2k$. But since $\sum_{n < k \le m} 2k > 0$, this is impossible.

b $\sum_{0 \le k \le n} 2k$ is not onto $\mathbb{Z}^+$. It is always even and $31 \in \mathbb{Z}^+$, so there does not exist an $n \in \{3, 4, 5, ...\}$ such that $\sum_{3 \le k \le n} 2k = 31$.

c $\sum_{3 \le k \le n} 2k$ is not one-to-one, so it can not be a bijection.

d $\sum_{3 \le k \le n} 2k = 2 \sum_{3 \le k \le n} k = 2\left(\sum_{0 \le k \le n} k - 3 \right) = 2\left(\frac{n(n+1)}{2} - 3 \right) = n^2 + n - 6$

4. ***a*** true We let $H(x)$ mean that *x* works hard in Discrete, $W(x)$ mean that *x* will do well in Discrete, and $G(x)$ mean that *x* will be admitted to grad school. The premises are

$\forall x\,(W(x) \to G(x))$	1
$\forall x\,(H(x) \to W(x))$	2
$H(Al)$	3
$H(Al) \to W(Al)$	4 Universal Instantiation on 2
$W(Al)$	5 Modus ponens on 3 & 4
$W(Al) \to G(Al)$	6 Universal Instantiation on 1
$G(Al)$	7 Modus ponens on 5 & 6

5. The statement is not a tautology. Consider the interpretation with one element, *Daisy*. Predicate *P* is satisfied for *Daisy* and predicate *Q* is not. That is, in the interpretation $P(Daisy)$ is true and $Q(Daisy)$ is false. Thus, for all x, $P(x) \to Q(x)$ is false and $Q(x) \to P(x)$ is true, so that $(P(x) \to Q(x)) \leftrightarrow (Q(x) \to P(x))$ is false.

6. The CONJECTURE is false If $A = \varnothing$ and $B = C = \{Fred\}$, then if $A \cup C = B \cup C = \{Fred\}$, although $A \neq B$.

7. ***a*** $f(n) = n + 1$

b

n : 0 1 2 3 4 5 6 7...

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$f(n)$: 0 1 2 3 4 5 6 7...

That is, $f(n) = \begin{cases} 0, \text{ if } n = 0 \vee n = 1 \\ n - 1, \text{otherwise} \end{cases}$