CS2022/MA2201

Midterm Exam

Name______________

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All documentation permitted

1. (20 points) For each of the following statements, state whether it is true or false. If a statement is false, then show why.

a $\neg(p \vee \neg q)$ and $(q \wedge \neg p)$ are logically equivalent.

b For any sets A, B and C, $A-(B\cap C)=(A-B)\cup(A-C)$.

c $f:\mathbb{Z}\to\mathbb{Z}$, where $f(n)=2n+1$, is one-to-one.

d $f:\mathbb{Z}\to\mathbb{Z}$, where $f(n)=2n+1$, is onto.

e For any sets A, B, C, if $A\subseteq B$ then $A\times C\subseteq B\times C$.

2. (10 points) Are each of the following arguments valid? Justify your response.

a

$$
\begin{array}{l}
p \rightarrow q \\
\neg p \\
\hline
\therefore \neg q
\end{array}
$$

b

$$
\begin{array}{l}
p \rightarrow r \\
q \rightarrow r \\
q \vee \neg r \\
\hline
\therefore \neg p
\end{array}
$$

3. (25 points) Prove or give a counterexample to the following

CONJECTURE: $3^n < n!$ for all $n \geq 7$.

4. (15 points) Which of the following statements are true in the interpretation in which the universe of discourse is $\mathbb{R}$, the real numbers, and $P(x, y)$ means $x + 2y = 5$.

a $(\forall x)(\exists y) P(x, y)$

b $(\exists x)(\forall y) P(x, y)$

c $(\exists x)(\forall y)(P(x, y) \vee x = y)$

5. (20 points) Is the set $\{10^n \mid n \in \mathbb{N}\}$ countably infinite? Justify your answer.

6. (10 points) Consider the function $f:\mathbb{Z}\rightarrow\mathbb{Q}$ where $f(n)=\frac{1}{n^2+5}$.

a Is *f* one-to-one?

b Is *f* onto?

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Solutions to Midterm Exam

1. ***a*** true

p	q	$\neg(p \vee \neg q)$	$(q \wedge \neg p)$
f	f	f	f
f	t	t	t
t	f	f	f
t	t	f	f

b true $A-(B\cap C)=A\cap\overline{(B\cap C)}=A\cap(\overline{B}\cup\overline{C})=(A\cap\overline{B})\cup(A\cap\overline{C})=(A-B)\cup(A-C)$

c true Since $(2n+1=2m+1)\to(n=m)$, *f* is one-to-one.

d false Since $(\forall n\in\mathbb{Z})$ *f*(*n*) is odd, $\neg(\exists n)\,f(n)=42$.

e true $A\times C=\{(x,y)\mid x\in A\wedge y\in C\}$. But since $A\subseteq B$, $x\in A\to x\in B$. Hence $(x,y)\in A\times C\to(x,y)\in B\times C$, which implies that $A\times C\subseteq B\times C$.

2. ***a*** The argument is not valid. If *p* is false and *q* is true, then the hypotheses are true but the conclusion is false.
b The argument is not valid. If *p*, *q* and *r* are all true, then the hypotheses are true but the conclusion is false.

3. The CONJECTURE is true. We will prove it by induction on *n*.
Basis: $P(7)$: $3^7=2187<5{,}040=7!$

Induction Hypothesis: $P(n)$ Assume that for fixed $n\geq 7$, $3^n<n!$.

Induction Step: $P(n)\to P(n+1)$ $3^{n+1}=3*3^n$, which, by the Induction Hypothesis, is less than or equal to $3*n!<(n+1)n!=(n+1)!$, which establishes the theorem.

4. ***a*** true ***b*** false ***c*** false

5. Certainly $\{10^n\mid n\in\mathbb{N}\}=\{1,10,100,1000,...\}=\{10^0,10^1,10^2,10^3,\ldots\}$ is infinite. A bijection $f:\{10^n\mid n\in\mathbb{N}\}\to\mathbb{N}$ is $f(m)=\log_{10}m$, so it is countably infinite.

6. ***a*** No, because $f(1)=f(-1)=\frac{1}{6}$.

b No, because $\neg(\exists n)\,f(n)=\frac{1}{7}$.