CS2022/MA2201

Midterm Exam

Name_____________

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All documentation permitted

1. (25 points) For each of the following statements, state whether it is true or false. If a statement is false, then give a counterexample.

a For any two functions $f:\mathbb{Z}\to\mathbb{Z}$ and $g:\mathbb{Z}\to\mathbb{Z}$, if *f* is $O(g)$ and *g* is $O(f)$, then it must be the case that *f*=*g*.

b For any sets *A*, *B* and *C*, $(A\cap B)\subseteq(A\cup C)\cap(B\cup C)$.

c Implication is associative, that is, $(p\to q)\to r\Leftrightarrow p\to(q\to r)$.

d The set of fractions $\left\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},...\right\}$ is countably infinite.

e For any set *A*, $P(A\times\varnothing)\subseteq P(A)$.

2. (25 points) The real numbers are denoted by $\mathbb{R}$, and rational numbers, $\mathbb{Q}$, are defined as $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z} \wedge q \in \mathbb{Z} - \{0\}\}$. The *irrational* numbers are the real numbers which are not rational. Note that $\mathbb{Q} \subseteq \mathbb{R}$.

a State the following proposition in predicate logic.

> The sum of any rational number and any irrational number is always irrational.

b Prove the proposition of part ***a***.

3. (25 points) Use mathematical induction to prove that $2n+5<3^n$ for all $n\geq 3$.

4. (25 points) ***a*** Give an example of a function $f:\mathbb{Z}\to\mathbb{Z}$ which is onto but not one-to-one.

b Give an example of a function $g:\mathbb{Z}\to\mathbb{Z}$ which is one-to-one but not onto.

c Give an example of a function $h:\mathbb{Z}\to\mathbb{Z}$ which is one-to-one and onto.

dGive an example of a function $\Phi:\mathbb{Z}\to\mathbb{Z}$ which is neither one-to-one nor onto.

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Solutions to Midterm Exam

1. ***a*** false Consider $f(n) = n$ and $g(n) = 2n$.
b true
c false Let *p* and *r* be false and let *q* be true.
d true In fact, the order in which the sequence is listing corresponds to a natural pairing.
e true For any set *A*, $A \times \varnothing = \varnothing$ so $P(A \times \varnothing) = \{\varnothing\}$ and $\varnothing \in P(A)$ (even if *A* is empty).

2. ***a*** $(\forall x)(\forall y)(x \in \mathbb{Q} \wedge y \in \mathbb{R} - \mathbb{Q}) \rightarrow x + y \in \mathbb{R} - \mathbb{Q}$
b Proof by contradiction.
We are given *x*=*p*/*q* and $y \in \mathbb{R} - \mathbb{Q}$. Assume *x*+*y* is rational, that is, $x + y \in \mathbb{Q}$. Because *x*+*y* is rational, there exist p_1 and q_1 such that $x + y = \frac{p}{q} + y = \frac{p_1}{q_1}$. Then we could solve for *y* as $y = \frac{p_1}{q_1} - \frac{p}{q} = \frac{p_1 q - p q_1}{q_1 q}$. This would mean that $y \in \mathbb{Q}$, which is a contradiction. Hence $x + y \in \mathbb{R} - \mathbb{Q}$.

3. *P*(3) As a basis, $2 * 3 + 5 = 11 < 27 = 3^3$.
$P(n) \rightarrow P(n+1)$ Fix $n \geq 3$ and assume that $2n + 5 < 3^n$.
$$2(n+1) + 5 = 2n + 5 + 2 < 3^n + 2 < 3^n + 2 * 3^n = 3^{n+1}$$

4. ***a*** $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$
b $g(n) = \begin{cases} n, \text{if } n < 0 \\ n+1, \text{if } n \geq 0 \end{cases}$ Note that $\neg(\exists n)\ f(n) = 0$
c $h(n) = n$
d $\Phi(n) = 42$