CS2022/MA2201

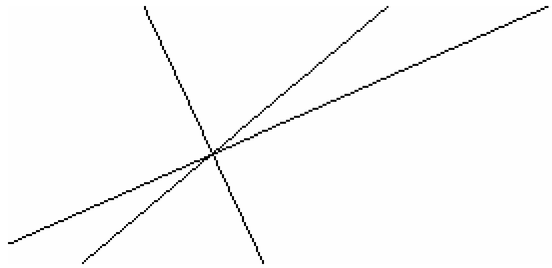
Final Exam

Name______________

<u>Date</u>: May 1, 2007
All documentation permitted

1. (20 points) Suppose you are given a list of integers, $(a_1, ..., a_n)$, such that $a_i \leq a_{i+1}$ for all $1 \leq i < n$. Describe an algorithm which uses $O(n)$ pairwise comparisons (tests of the form "Is $a_j \stackrel{?}{=} a_k$") in the worst-case to test if any integer appears twice in the list.

2. (15 points) Prove by induction that for any $n \geq 1$ lines in Euclidean 2-space, if the lines all pass through the same point and no pair of lines overlap, then they form $2n$ regions. For example, the following 3 lines form 6 regions.



3. (20 points) A deck of 52 playing cards has 4 aces.
a If 3 cards are randomly exposed with replacement (after exposure, the card is not removed and may be exposed again), what is the probabilty that all 3 cards are aces?

b If 4 cards are randomly exposed with replacement (after exposure, the card is not removed and may be exposed again), what is the probability that the first 2 cards are a pair **and** the last 2 cards are a pair? The pairs may or may not be the same. As examples, if all 4 cards are kings then this is a success, if the first 2 cards are kings and the second two cards are queens then this is a success, but if the first and last cards are kings and the second and third cards are queens then this is a failure.

4. (15 points) Tell which of the following relations are equivalence relations. If a relation is not an equivalence relation, tell which properties (reflexivity, symmetry, transitivity) it violates.

a R defined on $\mathbb{Z}^+$ such that aRb means that a ends in the same digit as b.

b R defined on $\mathbb{Z}$ such that aRb means that $ab \le 0$.

c R defined on $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that $(a,b)R(c,d)$ means that $ad = bc$.

5. (20 points) ***a*** Draw all the non-isomorphic simple graphs on 4 vertices which have a Hamilton circuit but no Euler circuit.

b Draw all the non-isomorphic simple graphs on 4 vertices which have an Euler circuit but no Hamilton circuit.

6. (10 points) For $n \geq 2$ let us say that an n-bit string is *good* if it has two consecutive 1's and n-2 0s. Note that for n=6 the strings 001100 and 000011 are good and 001010 and 011001 are not good. How many good n-bit strings are there?

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Solutions to <u>Final Exam</u>

1. $k \leftarrow 0$
for $j \leftarrow 1$ **to** n-1 **do if** $a_j = a_{j+1}$ **then** $k \leftarrow j$
if k>0 **then** "There is a duplicate" **else** "There is no duplicate"

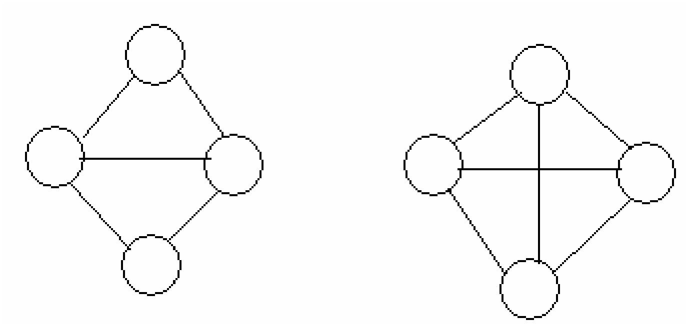
2. We prove the claim by induction on n. As a basis, there is 1 line forms 2*1 regions. We assume that n such lines form 2n regions. If n=1 then adding a second line divides the regions on both sides of the first line into 2 regions each, and 2n=2*2=2+2. If n>1, then an $(n+1)^{st}$ line lies in the cone formed by 2 of the previous lines, dividing the two regions through which it passes into 4 regions. And the number of regions is now 2n+2=2(n+1).

3. ***a*** $\left(\frac{1}{13}\right)^3 \approx .000455166$

b Whatever the first and third cards are, the probability that the second card matches the first is $\frac{1}{13}$, as is the probability that the fourth card matches the third. Since these probabilities are independent. the answer is $\left(\frac{1}{13}\right)^2 \approx .005917159763$.

4. ***a*** R is an equivalence relation.
b R is not an equivalence relation because it violates reflexivity and transitivity.
c R is an equivalence relation.

5. ***a***



b There are none.

6. n-1, because there are n-1 places to put the first 1. Once we know the position of the first 1, the contents of the other n-1 positions are determined.