Midterm Exam

Name________

Date: September 22, 1997
All documentation permitted

1. (5 points) Is \(\neg q \lor p \iff (q \rightarrow p)\)? Justify your response.

2. (12 points) Letting the Universe of Discourse be the set of students at WPI, let \(P(x)\) denote the statement “\(x\) likes CS2022/MA2201”. Express, in predicate calculus, the sentence “At least two students like CS2022/MA2201, though not everybody likes it”.

3. (12 points) Suppose that \(|A|=m=1\) and \(|B|=n=1\). What is the most that can be said about the relationship between \(m\) and \(n\) for each of the following to be true?
   a) There is an injection from \(A\) to \(B\).
   b) There is a surjection from \(A\) to \(B\).
   c) There is a bijection from \(A\) to \(B\).
4. (21 points) A) Translate the following inference into propositional logic.
   If today is Thursday, then I have a test in CS or a test in Econ. If my Econ professor is sick, then I will not have a test in Econ. Today is Thursday and my Econ professor is sick. Therefore I have a test in CS.

B) Is the inference correct? Justify your response.
5. (30 points) An irrational number is a number which is not rational, that is, it can not be expressed as the ratio of two integers. Prove or give a counterexample to the following: 

**CONJECTURE:** For all rational numbers \( x, x \neq 0 \), and irrational numbers \( y \), the product of \( x \) and \( y \) must be irrational.

(Hint: The conjecture states that it’s not possible for there to be a rational number \( x \times y \). If the CONJECTURE is true, then you must prove the no such rational number exists. Think of how we prove a negative result like “such a rational number can’t exist”. If it’s false, you only need demonstrate a rational \( x \neq 0 \) and an irrational \( y \) such that \( x \times y \) is rational.)
6. (20 points) Express the sum $\sum_{k=0}^{n} x^k$ using $O$-notation, in the form $O(x^m)$, where $m$ is the smallest integer for which the statement is true.
1. A truth table establishes the logical equivalence.

<table>
<thead>
<tr>
<th>q</th>
<th>p</th>
<th>¬q</th>
<th>(¬q ∨ p)</th>
<th>(q → p)</th>
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2. (∃x)(∃y)(P(x) ∧ P(y) ∧ (x ≠ y)) ∧ ¬(∀x)P(x)

3. 
   a) m=n
   b) m=n
   c) m=n

4. t - Today is Thursday.
   c - I have a test in CS.
   e - I have a test in Econ.
   s - My econ professor is sick.

   The argument is:
   t → (c ∨ e)
   s → ¬e
   t ∧ s

   ∴ c

   From t ∧ s we can conclude t and s (simplification). From t and t → (c ∨ e) we can conclude c ∨ e (modus ponens). From s and s → ¬e we can conclude ¬e (modus ponens). From c ∨ e and ¬e we can conclude c (disjunctive syllogism).

5. The conjecture is true. (Proof by Contradiction) If it were false, there’d be a rational x=a/b, a,b∈Z and an irrational y such that there exist c,d∈Z such that x*y is rational, that is, x*y=a*y/b=a*c/d. But this implies y=b*c/(a*d). But since b*c∈Z and a*d∈Z, this would imply that y is a rational number, which is a contradiction.

6. \[ \sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1} = O(x^n) \]