CS2022/MA2201
HW#5

DUE: Tuesday, April 10

1. (6 points) Let $(a_1, ..., a_n)$ be a sorted list of numbers. That is, $\forall i, 1 \le i < n, a_i \le a_{i+1}$.

a Give an algorithm with execution time in $O(1)$ to find a number not in the list. That is, your algorithm should return *x* such that $x \ne a_i$ for all $1 \le i \le n$.

b Give an algorithm with execution time in $O(n)$ to find a closest distinct pair of numbers in the list. That is, your algorithm should return $a_i, a_j, i \ne j$ such that $|a_i - a_j| \le |a_k - a_l|$ for all $k \ne l$.

2. (8 points) Consider the function λ which associates with each finite bit string (possibly empty string of 0's and 1's) the number of bits in the string. For example, $\lambda(001110) = 6$, and λ of the empty string is 0. Justify your answers to parts ***b***, ***c*** and ***d***.

a What are the domain and codomain of λ?
b Is λ one-to-one?
c Is λ onto?
d Is λ a bijection?

3. (6 points) For each of the following sets, tell whether it is finite, countably infinite or uncountably infinite.

a $\{x \in \mathbb{Q} \mid 17.2 < x \le 42\}$
b The set of all English sentences with no more than 1050 words.
c $\{x \in \Re \mid 2100 < x\}$

4. (5 points) Prove that $n^n > n!$ for all integers $n \ge 2$. The factorial function used in *n*! is defined on page 185 of our text.

5. (8 points) A set of lines is in *general position* if there do not exist three of the lines which pass through any point and no two of the lines are parallel to each other. Prove that for any $n \ge 3$ lines in the plane in general position, at least one of the regions they form is a triangle (bounded by exactly three lines).

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HW#5 SOLUTIONS

1. ***a*** **return** a_n+1

b Of all the adjacent pairs of points seen so far, the closest pair will be (a_s, a_{s+1})

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s ← 1
for i ← 2 to n−1 do
    if a_{s+1} − a_s > a_{i+1} − a_i then s ← i
return (a_s, a_{s+1})
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2. ***a*** The domain of λ is the set of all bit strings, and the codomain is the set of all nonnegative integers, or natural numbers, $\mathbb{Z}^+ \cup \{0\} = \mathbb{N}$.

b λ is not one-to-one since $\lambda(01) = \lambda(10) = 2$ although $01 \neq 10$.

c λ is onto since for each $n \in \mathbb{Z}^+ \cup \{0\}$, $\lambda(1^n) = n$, where 1^n is the string of n 1's, and 1^0 is the empty bit string.

d Because λ is not one-to-one, it is not a bijection.

3. ***a*** countably infinite ***b*** finite ***c*** uncountably infinite

4. We prove this by induction on *n*. For the basis, *n*=2, we note that $2^2 = 4 > 2! = 2$. For the inductive step, we assume that $n^n > n!$. It follows that

$$(n+1)^{n+1} = (n+1)*(n+1)^n > (n+1)*n^n > (n+1)*n! = (n+1)!$$

where the second inequality follows from the inductive hypothesis. By induction it follows that $n^n > n!$ for all integers $n \geq 2$.

5. The proof is by induction on *n*.

Basis: Since any three lines in general position can not meet in one point, they must meet pairwise in three points. The region inside the three line segments between these three intersection points forms a triangle.

Induction Hypothesis: Assume that for some *n*≥3, any placement of *n* lines in the plane in general position forms at least one triangle. Consider any placement of *n*+1 lines in the plane, and let *l* be any one of the lines. By the induction hypothesis, the other *n* lines form at least one triangle. Let τ be such a triangle. If *l* does not intersect τ, then τ is still a triangle after the placement of the *n*+1 lines. If *l* does intersect τ, then it intersects two of τ's lines in τ (call them l_1 and l_2). The region with the boundary line segments of l, l_1 and l_2 within the region of their pairwise intersection must be a triangle.