CS2022/MA2201
HW#4

DUE: Monday, September 22

1. (5 points) ALGORITHM 4 of our text is the following algorithm to BUBBLESORT a list of n numbers $a_1,\ldots,a_n$:

    procedure BUBBLESORT($a_1,\ldots,a_n$)
    for $i \leftarrow 1$ **to** n-1
        for $j \leftarrow 1$ **to** n-i
            if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}

As a function of n, how many times is the comparison $a_j > a_{j+1}$ executed**?**

2. (8 points) Find a closed form for the following summation

$$\sum_{2\le i\le n}\left(\left(\frac{2}{5}\right)^i - i + 3\right).$$

3. (9 points) Tell whether each of the following statements is true or false, and justify your response.

(a) 2^n is $O\left(2^{n+10}\right)$.

(b) 2^{n+10} is $O\left(2^n\right)$.

(c) 2^{2n} is $O\left(2^n\right)$.

4. (5 points) For what values of $n \in \mathbb{Z}^+$ is $n! < n^n$? Prove your answer.

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HW#4 Solutions

1. $\sum_{1\le i\le n-1}\sum_{1\le j\le n-i} 1 = \sum_{1\le i\le n-1}(n-i) = \sum_{1\le i\le n-1} n - \sum_{1\le i\le n-1} i = n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}$

2.
$$\sum_{2\le i\le n}\left(\left(\frac{2}{5}\right)^i - i + 3\right) = \sum_{2\le i\le n}\left(\frac{2}{5}\right)^i - \sum_{2\le i\le n} i + \sum_{2\le i\le n} 3 = \left(\sum_{0\le i\le n}\left(\frac{2}{5}\right)^i - \left(\frac{2}{5}\right)^1 - \left(\frac{2}{5}\right)^0\right) - \left(\sum_{1\le i\le n} i - 1\right) + 3\sum_{2\le i\le n} 1$$
$$= \frac{1-\left(\frac{2}{5}\right)^{n+1}}{1-\frac{2}{5}} - \frac{7}{5} - \frac{n(n+1)}{2} + 1 + 3(n-1) = \frac{5}{3}\left(1-\left(\frac{2}{5}\right)^{n+1}\right) - \frac{n^2}{2} + \frac{5n}{2} - \frac{17}{5}$$
$$= \frac{5n}{2} - \frac{n^2}{2} - \frac{5}{3}\left(\frac{2}{5}\right)^{n+1} - \frac{26}{15}$$

3. (a) 2^n is $O\left(2^{n+10}\right)$. Choose $c=1$ and $n_0=1$ and for all $n>n_0$, $2^n \le 2^{n+10} = 1024*2^n$

(b) 2^{n+10} is $O\left(2^n\right)$. Choose $c=1024$ and $n_0=1$ and for all $n>n_0$, then $2^{n+10} = 1024*2^n \le c2^n = 1024*2^n$.

(c) 2^{2n} is not $O\left(2^n\right)$. If it were, there would exist constants c and n_0 such that $2^{2n} \le c2^n$ for all $n>n_0$. But dividing both sides of the inequality would imply that for all n sufficiently large, $2^n \le c$. Since this is impossible for any constant c, it follows by contradiction that 2^{2n} can not be $O\left(2^n\right)$.

4. **<u>Theorem</u>**: $(\forall n \ge 2)\ n! < n^n$.

<u>Proof</u>: Basis: $P(2)$ $2!=2<2^2=4$

Induction Hypothesis: Assume that for $n \ge 2$ that $n! < n^n$.

Induction Step: $(n+1)! = (n+1)n!$, and by the Induction Hypothesis,

$(n+1)n! < (n+1)n^n < (n+1)(n+1)^n = (n+1)^{n+1}$, which establishes the Theorem.