CS2022/MA2201
HW#3 SOLUTIONS

1. ***a*** yes (every number has a unique decomposition into primes)
b no There do not exist integers *x* and *y* such that $f(x,y)=2^x 3^y=11$.
c no $16=2^4 4^0 = f(4,0)=2^2 4^1 = f(2,1)=2^0 4^2 = f(0,2)$
b no There do not exist integers *x* and *y* such that $f(x,y)=2^x 4^y=11$.

2. The CONJECTURE is false. Let $f(x)=2x$ and $g(x)=x+5$. Since $g\circ f(x)=2x+5$ and $f\circ g(x)=2x+10$, it follows that $g\circ f(x)$ **never** equals $f\circ g(x)$ (unless 5=10).

3. The CONJECTURE is true. The rational number $\frac{p_1q_2+p_2q_1}{2q_1q_2}$ satisfies $\frac{p_1}{q_1}<\frac{p_1q_2+p_2q_1}{2q_1q_2}<\frac{p_2}{q_2}$ (in fact it's halfway between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$) and then we can continue the same process to produce an unbounded number of rationals between $\frac{p_1}{q_1}$ and $\frac{p_1q_2+p_2q_1}{2q_1q_2}$.

4. ***a*** $f(n)=\sum_{i=4}^{n}\sum_{k=1}^{i-2}2=\sum_{i=4}^{n}2\sum_{k=1}^{i-2}1=\sum_{i=4}^{n}2(i-2)=2\sum_{i=4}^{n}i-4\sum_{i=4}^{n}1$. Before evaluating the sum further, we note that $f(1)=f(2)=f(3)=0$. For $n\ge 4$,

$$f(n)=2\left(\sum_{i=1}^{n}i-\sum_{i=1}^{3}i\right)-4(n-3)=2\left(\frac{n(n+1)}{2}-6\right)-4n+12=n^2-3n$$

Defined over the entire domain,

$$f(n)=\begin{cases}0,\text{if } 1\le n\le 3\\ n^2-3n,\text{if } 4\le n\end{cases}$$

b $\sum_{i=2}^{n}\left(\frac{3}{7}\right)^i=\sum_{i=0}^{n}\left(\frac{3}{7}\right)^i-\sum_{i=0}^{1}\left(\frac{3}{7}\right)^i=\frac{1-\left(\frac{3}{7}\right)^{n+1}}{1-\frac{3}{7}}-\frac{10}{7}=\frac{9}{28}-\frac{7}{4}\left(\frac{3}{7}\right)^{n+1}$