Binary Trees

Objectives:

- to define the binary tree ADT
- to present particular types of binary trees
- to describe different implementations of binary trees and analyze those implementations
- to present some applications of binary trees

Definitions and Properties

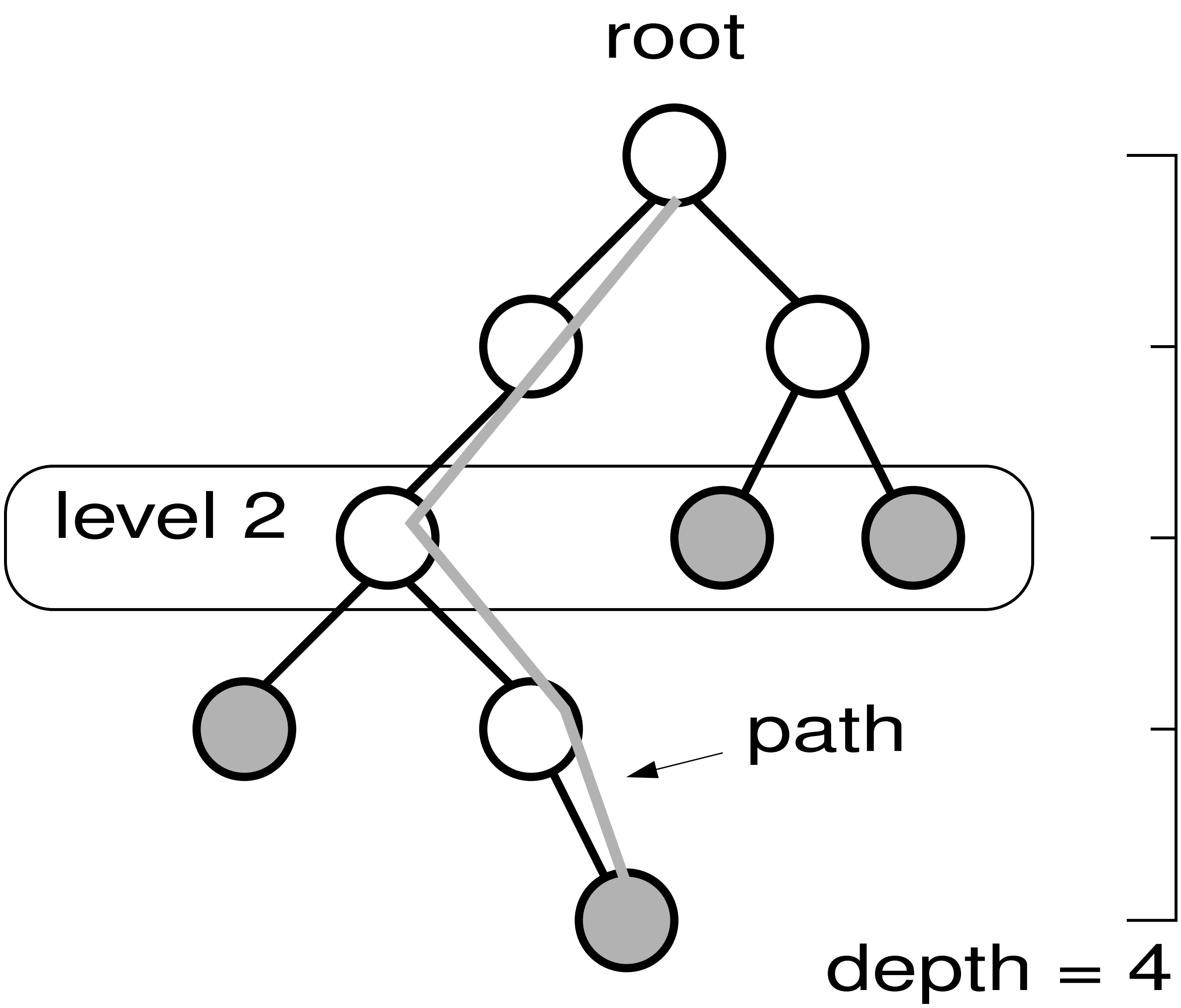
Definition. A *binary tree* is a finite set of nodes connected by edges such that:

- the set is either empty or consists of a node called the **root** and two binary trees, called the left and right **subtrees**, which are disjoint from each other and the root;
- there is an **edge** from the root to the roots of its subtrees.

Terminology:

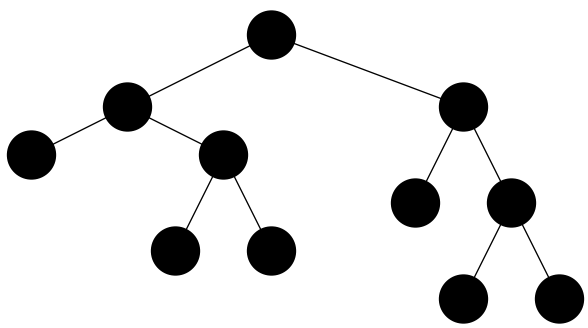
- **children** of a node - the roots of its subtrees (a node is called parent of its children)
- **path** - a sequence of nodes $n_1, n_2, \ldots, n_k$ such that n_i is the parent of n_{i+1} for all $0<i<k$
- **length** of a path - the number of edges along it
- a node *A* is an **ancestor** of another node *B* if there is a path from *A* to *B* (*B* is a descendant of *A*)
- **depth** of a node - the length of the path from the root to that node

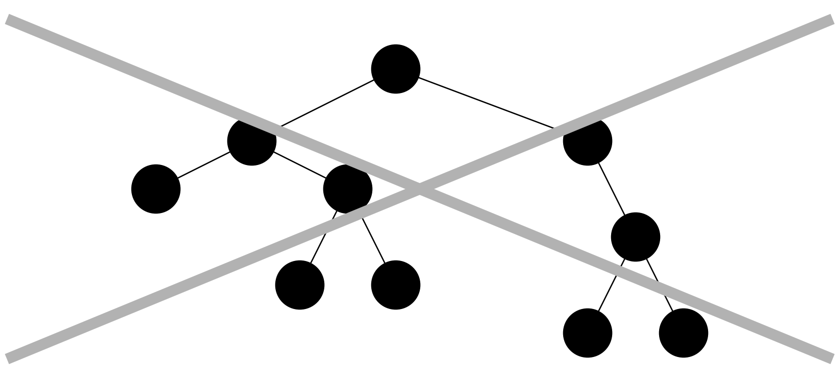
- **height** of a tree - one more than the length of its deepest node
- **level** - the set of nodes of a given depth
- **leaf node** - has both its children empty
- **internal node** - has at least one non-empty child


root
level 2
path
depth = 4

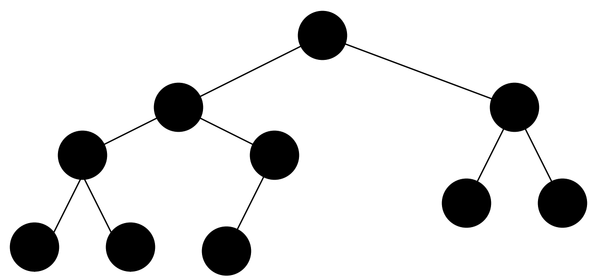
Special binary trees

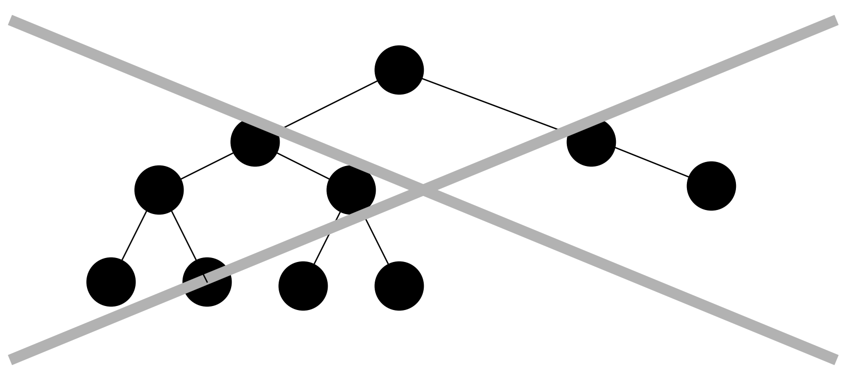
full binary tree: each node is either a leaf, or an internal node and has exactly two non-empty children





complete binary tree: all the levels are completely filled, excepting possibly the deepest one, which is filled from the left to the right





Characterization of binary trees

Theorem. The number of leaves in a non-empty full binary tree is one more than the number of internal nodes.

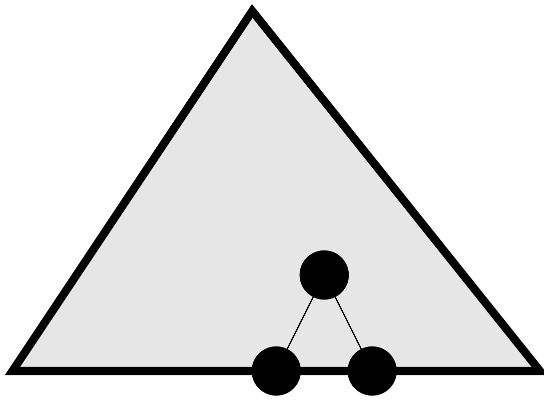
Proof. By mathematical induction on the number of internal nodes.

1. **Base:**

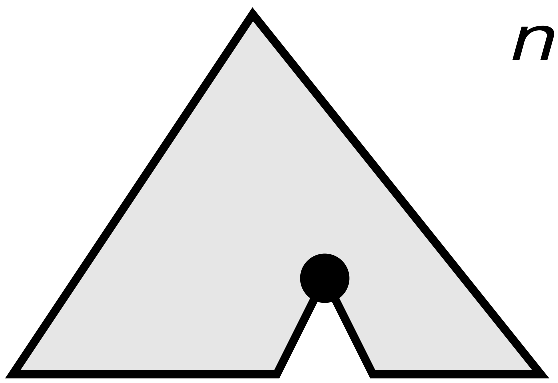
$n=0$ $n=1$

2. **Induction hypothesis:** assume that a full binary tree containing $n-1$ internal nodes has n leaves

3. Induction step:

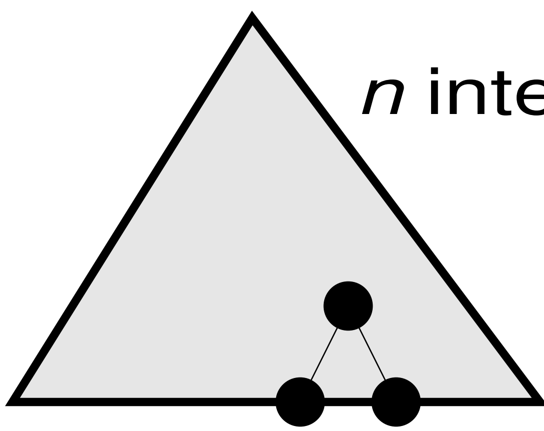


n internal nodes



n-1 internal nodes

n leaves



n internal nodes

&

n+1 leaves

q.e.d

Theorem. The number of empty subtrees in a non-empty binary tree is one more than the number of nodes in the tree.

Proof.

- every node of a binary tree has two children, for a total of $2n$ children for a binary tree of n nodes.
- every node, except for the root node has one parent, for a total of $n-1$ parents ($n-1$ nonempty children)
- results that the remaining $n+1$ children ($2n-(n-1)$) must be empty

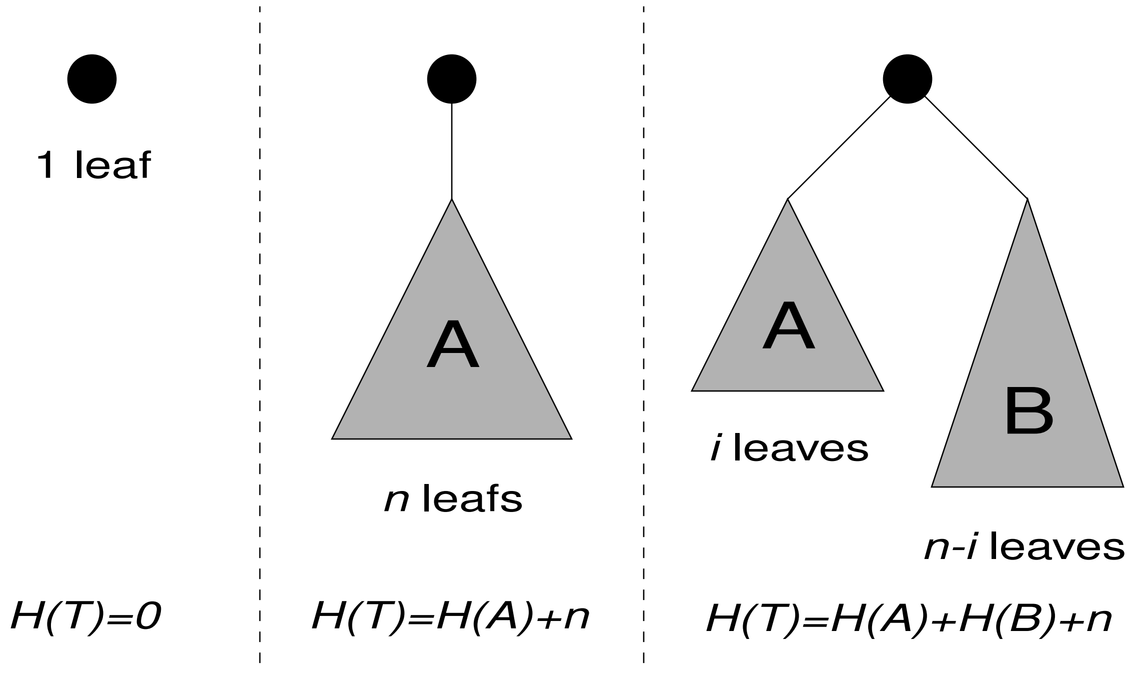
Theorem. Any binary tree with *n* leaves has an average height of at least lg*n*.

Proof.

- let *T* be a binary tree with n leaves and denote by *H(T)* the sum of depths of the leaves
- we need to show that for a binary tree with *n* leaves

$$H(T) \geq n \log n$$

- the root of T can have 0,1 or 2 children



- let *h(n)* be the smalles value for *H(T)* when *T* is a binary tree with n leaves (define *h(0)=0*)

$$h(n) = min(h(i) + h(n-i) + n)$$

- since *i=0* and *i=n* cannotyield the minimum $0<i<n$

- prove by mathematical induction that $h(n)) \geq n \log n$ for all $n \geq 1$.

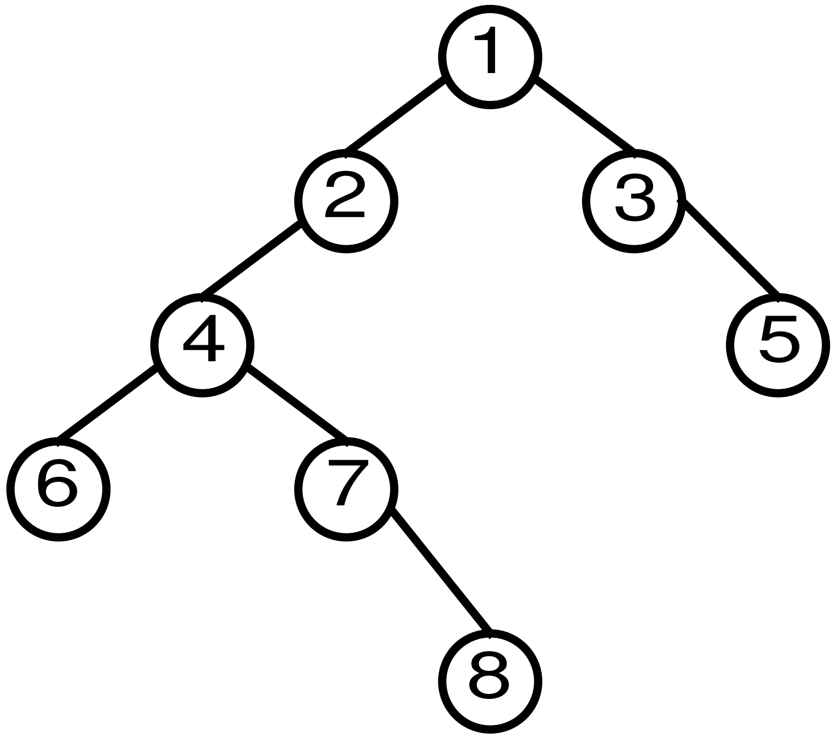
Binary tree traversals

- **Traversal:** processing all the nodes in a systematic manner

Preorder traversal: visit the node first, then traverse its left subtree in preorder, then its right subtree in preorder

Postorder traversal: traverse the left subtree in postorder, then the right subtree in postorder, and finally visit the node itself

Inorder traversal: traverse the left subtree in inorder, then visit the node, and finally traverse the right subtree in inorder



- preorder: 1,2,4,6,7,8,3,5
- postorder: 6,8,7,4,2,5,3,1
- inorder: 6,4,7,8,2,1,3,5

The binary tree ADT

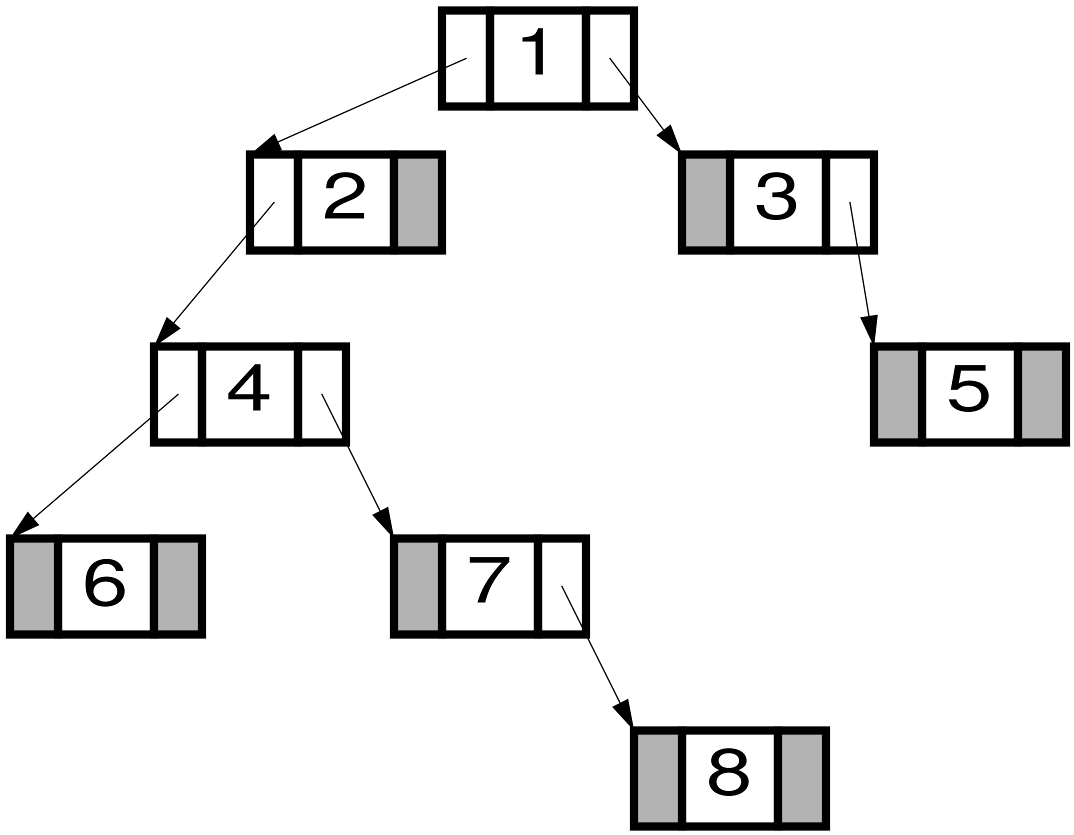
Type: a set of binary trees

Operations:

- insert a node
- remove a node
- find a node

Binary tree implementations

1. **Pointer-based** implementation

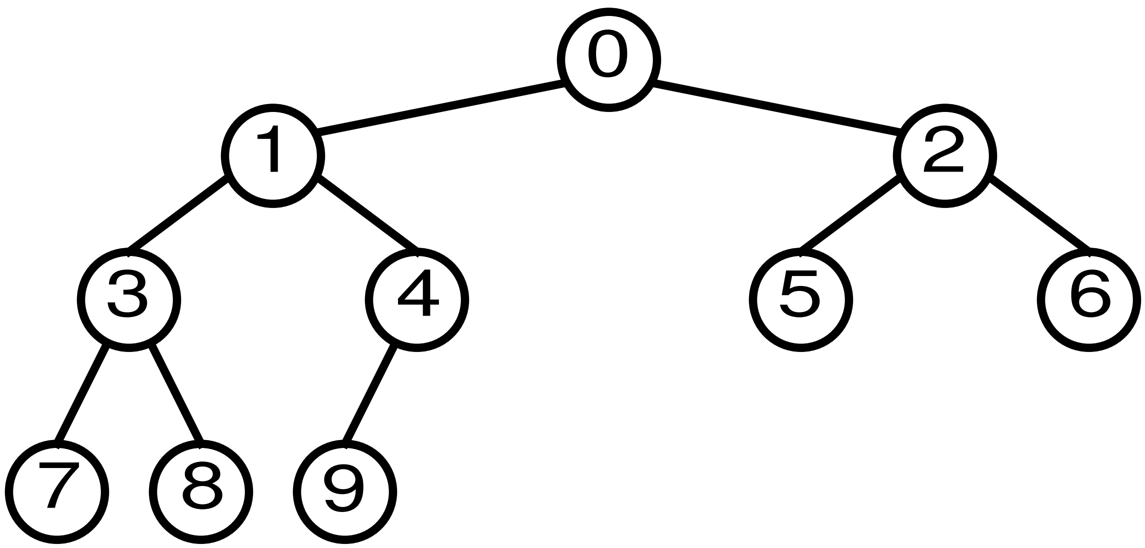


```
class BinNode{
public:
      Element value;
      BinNode*       left;
      BinNode*       right;
      BinNode(){left = right = NULL;}
      BinNode(Element e,
                  BinNode* l=NULL,
                  BinNode* r=NULL){
            value = e; left = l; right = r;
      }
      ~BinNode() {};
};
```

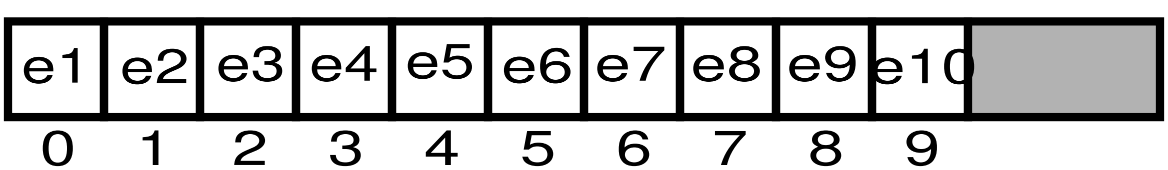
- **space requirement:** *n(2p+d)*, but can be reduced by eliminating pointers from leaf nodes (*is this really true?*)

2. **Array-based implementation** of complete binary trees

- number the nodes level by level from the left to the right:



- place the elements stored in the node in an array, at the index corresponding to the numbering



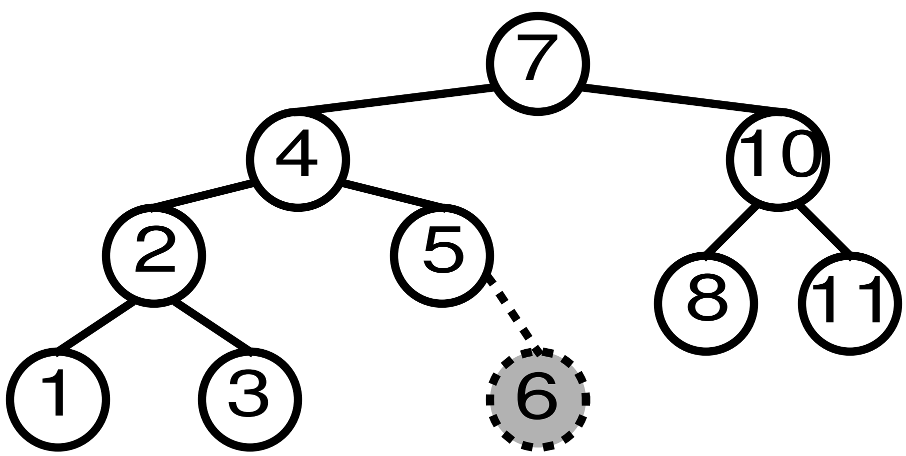
Calculating the indicex of relatives of a node:

- parent(r)=(r-1)/2, if 0<r<n
- left(r)=2r+1 if 2r+1<n
- right(r)=2r+2 if 2r+2<n
- left-sib(r)=r-1 if r -even, 0<r<n
- right-sib(r)=r+1 if r -odd, r+1<n

Binary search trees

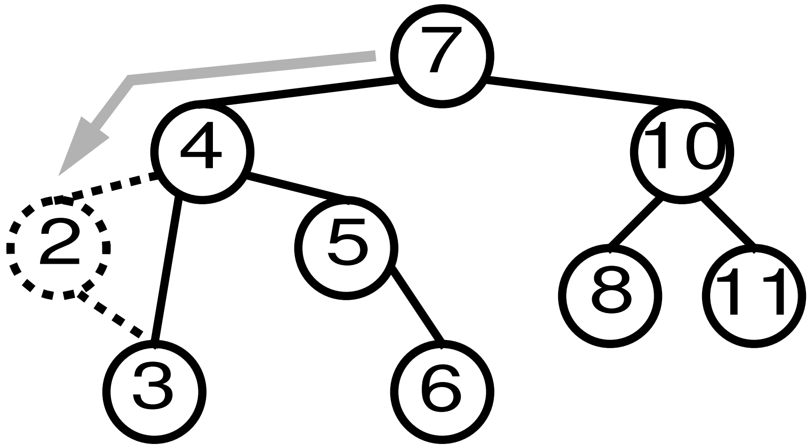
Definition. A **binary search tree** (BST) is a binary tree in which all nodes of the left subtree of a node store values less than the value stored in that node, and all nodes of the right subtree of a node store values greater than the values stored in that node.

- **inserting** a value into a BST

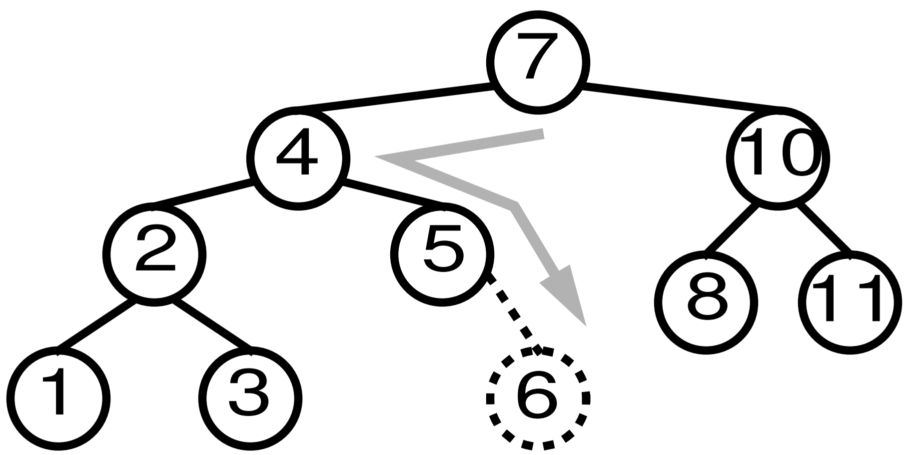


- **deleting** a node from a BST

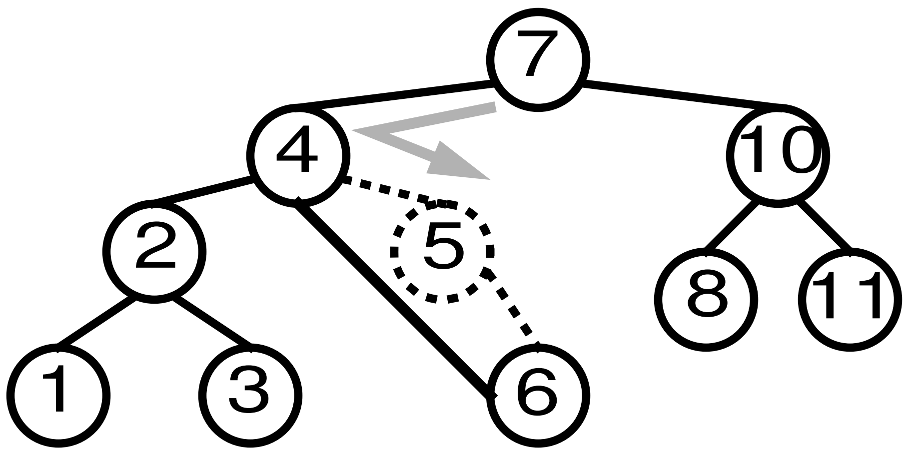
1. deleting the minimal value



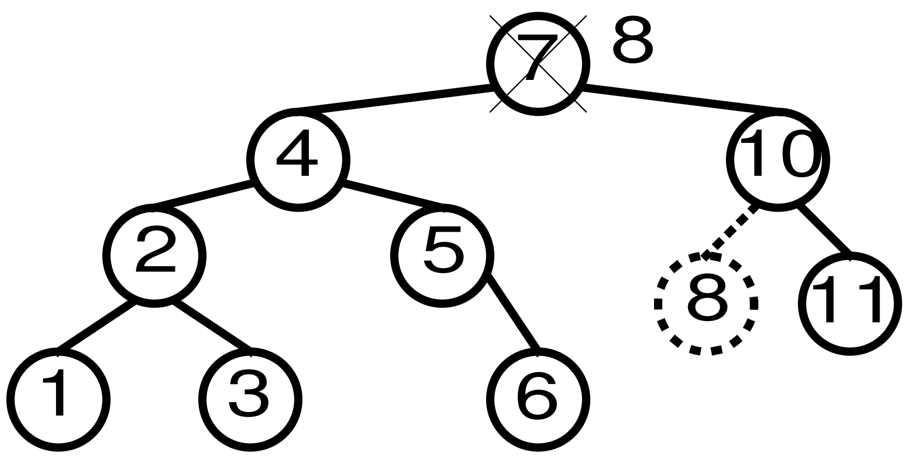
2. deleting an arbitrary node



no children



one child



two children

The **major problem** with the BST is that its shape depends on the order in which nodes are inserted.